CAUCHY COMPLETION IN ENRICHED CATEGORIES

by

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Statement

The following statement by Professor Ross Street explains the format of this thesis and the circumstances of its submission.

Before his death, it was planned that Scott Johnson's thesis would consist of the solutions to three problems concerning Cauchy completion in category theory. The Cauchy completion of an enriched category was defined by F.W. Lawvere in a paper which developed enriched category theory on the paradigm of metric spaces. However, the concept is basic to Morita equivalence in the context of additive categories where it is also called the Karoubi envelope.

The first of the three problems concerned a Morita-type theorem for monoidal categories. Johnson solved this problem by identifying the appropriate notion of Cauchy completion for monoidal enriched categories. He published the following paper which includes this result and some related matters:

S.R. Johnson, "Monoidal Morita equivalence", *Journal of Pure & Applied Algebra* **59** (1989) 169-177.

The second problem concerned the question of whether the Cauchy completion of a small enriched category was small. It had been conjectured that this was the case when the base for enrichment was a suitably algebraic category; a counterexample was known to the general question. Johnson solved this problem with a theorem that the Cauchy completion of any small enriched category is small provided the base monoidal category is locally finitely presented as a mere category. He also threw light on the nature of the counterexample. Johnson lectured on this work at the Isle of Thorns (Sussex, England); it was extremely well received. He is well remembered by the international community because of that talk. Just before his death he told me of the extension of this result to categories enriched over a bicategory: indeed, I found the details of this in his papers after his death. (This interest in bicategories came after his work on the third problem described below.) I also found a preprint of a paper on the result for a monoidal base prepared (equipped with covering letter) for submission to the Journal of Pure and Applied Algebra. I believe he had been holding off submission in order to seek my opinion as to whether to publish this or the more general result with a bicategory as base. It was my opinion that the generalisation was fairly routine for those who were interested, and that the essential idea was already present in the monoidal case which would have a wider audience. I decided to submit the preprint prepared by Johnson without change, and it was accepted without change. It appears as:

S.R. Johnson, "Small Cauchy completions", *Journal of Pure and Applied Algebra* **62** (1989) 35-45.

Now I come to Johnson's third problem. This came from work of Aurelio Carboni and myself. I had characterized those bicategories M which have enriched categories as objects and modules as arrows. Bicategories of Cauchy complete enriched categories and enriched functors between them were thus characterized as the subbicategories H of such M consisting of the arrows with right adjoints. Our problem was to characterize such H directly, without recourse to M. This is somewhat like having a characterization of the category of sets with relations as arrows, and seeking a characterization of the category of sets with functions as arrows. So we sought axioms on a bicategory H such that it could be embedded in such an M as the subbicategory of arrows with right adjoints. Carboni and I had some axioms which were sufficient to construct M but not with all the desired properties. The difficulty had to do with colimits: coproducts and coequalizers. That was the state of the problem when I turned it over to Johnson.

In the second half of 1988, Johnson discovered a remarkable new axiom. He spoke of this in the Sydney Category Seminar, thus establishing his priority on the idea. At first he was claiming that this axiom solved the difficulty with coproducts. Then, on the day before his death, he told me over the telephone that he could see how to deal with coequalizers. I told him this was good news.

After Scott Johnson's death, I worked (with the aid of his handwritten notes) on trying to reproduce his work. I soon succeeded, using his axiom, with the details needed for coproducts. I typed up the work to this point and showed it to various people. Each time I went back to the problem I made some progress, but could not finish it off.

In the August 1992, Dr Dominic Verity (Research Fellow, Macquarie University) solved the problem concerning coequalizers. This vindicated Johnson's claim. Verity made some other improvements, and prepared the preprint:

> Aurelio Carboni, Scott Johnson, Ross Street and Dominic Verity, "Modulated bicategories".

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Originality

The papers

> S.R. Johnson, Monoidal Morita equivalence, Journal of Pure & Applied Algebra 59 (1989) 169-177.
>
> S.R. Johnson, Small Cauchy completions, Journal of Pure & Applied Algebra 62 (1989) 35-45.

are totally Scott Johnson's original work.

In the paper

> Aurelio Carboni, Scott Johnson, Ross Street and Dominic Verity, "Modulated bicategories"

Scott Johnson's fundamental axiom occurs as 5.2 (vi) on page 24. The application of this axiom is in the proof of Theorem 5.18 (middle of page 33). The main results 5.20, 5.21, 5.22 of that important Section 5 all depend on Theorem 5.18.

The other aspect of the axiom is that it should be true in the examples. It is, of course, by studying the examples, that Johnson discovered possible axioms. The verification that the examples satisfy his axiom is at the end of the proof of Theorem 6.18 on page 49.

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MONOIDAL MORITA EQUIVALENCE

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Morita equivalence has been studied for categories enriched over a monoidal category. For such enriched categories *themselves* with a monoidal structure, we define a *monoidal* Cauchy completion, and derive many of the Morita theorems in this context. Conditions under which the monoidal Cauchy completion is closed are also discussed.

1. Introduction

After the work of Morita [13], concerned with the equivalence of the categories R-Mod and S-Mod for rings R and S, many have studied Morita theory in the context of categories enriched over a monoidal category $\mathscr{V}$ or even over a bicategory. See, for example [5, 6, 10, 11, 15, 16].

A summary of results known prior to 1981 can be found in [3]. In particular, Lawvere [10] defined the Cauchy completion $\mathscr{Q}\mathscr{A}$ of a $\mathscr{V}$-category $\mathscr{A}$, generalising the Cauchy completion of a metric space (the case $\mathscr{V} = \mathbb{R}^+$) and the idempotent-splitting completion of an ordinary category ($\mathscr{V} = \mathbf{Set}$). Lindner [11] then showed that $\mathscr{V}$-categories $\mathscr{A}$ and $\mathscr{B}$ are Morita equivalent ($[\mathscr{A}, \mathscr{V}] \simeq [\mathscr{B}, \mathscr{V}]$) precisely when their Cauchy completions are equivalent.

If we consider $\mathscr{V}$-categories with a *monoidal* structure, the questions arise whether there is a corresponding *monoidal* Cauchy completion, and whether standard Morita theorems are valid in the monoidal setting. Im and Kelly [7] have studied the free monoidal cocompletion $\mathscr{P}\mathscr{A}$ of a small monoidal $\mathscr{V}$-category $\mathscr{A}$, and much of their work extends easily to free monoidal $\mathscr{F}$-cocompletions where $\mathscr{F}$ is any set of weights for colimits. This, together with the observation of Street [14] that the Cauchy completion is just the free cocompletion under absolute colimits, gives us a monoidal structure on the Cauchy completion $\mathscr{Q}\mathscr{A}$ of any small monoidal $\mathscr{A}$.

From the principle that a monoidal functor is a monoidal equivalence if and only if it is strong (that is, preserves the monoidal structure to within isomorphism) and has an underlying functor which is an equivalence, it will follow that there is a monoidal equivalence $\mathscr{P}\mathscr{A} \simeq \mathscr{P}\mathscr{Q}\mathscr{A}$ for any monoidal $\mathscr{A}$. It can then be shown that much of Morita theory carries over to the monoidal case.

In [4], Day showed that the monoidal $\mathscr{P}\mathscr{A}$ is biclosed for any small monoidal $\mathscr{A}$. It is far from being true that $\mathscr{Q}\mathscr{A}$ is always biclosed (in fact this implies that the tensor product of $\mathscr{A}$ preserves colimits in both variables), but if $\mathscr{A}$ is *near* closed, in a sense to be made precise, the internal hom of $\mathscr{P}\mathscr{A}$ will restrict to $\mathscr{Q}\mathscr{A}$ making $\mathscr{Q}\mathscr{A}$ closed.

2. Preliminaries

All of the results here apply to categories enriched over a complete and cocomplete symmetric monoidal closed category $\mathscr{V}$ with unit I and tensor product $\otimes$. A $\mathscr{V}$-functor will be called a functor if it is understood that the domain and codomain are $\mathscr{V}$-categories. Similarly a $\mathscr{V}$-natural transformation between $\mathscr{V}$-functors will be called simply a natural transformation. For $\mathscr{V}$-categories $\mathscr{A}$ and $\mathscr{B}$, $[\mathscr{A},\mathscr{B}]$ will denote the $\mathscr{V}$-functor category and $[\mathscr{A},\mathscr{B}]_0$ will denote its underlying ordinary category of functors from $\mathscr{A}$ to $\mathscr{B}$ and natural transformations between them. If $\mathscr{A}$ is not small, $[\mathscr{A},\mathscr{B}]$ may only exist as a $\mathscr{V}'$ category for some extension $\mathscr{V}'$ of $\mathscr{V}$ as in [9, Section 3.11]. We use $Acc[\mathscr{A},\mathscr{V}]$ to denote the $\mathscr{V}$-category of *accessible* functors: those that are left Kan extensions of some $\mathscr{K}\to\mathscr{V}$ with $\mathscr{K}$ small (see [11], where such functors are called *small*; the term *accessible* is that used in [9] and [1]; of course every functor $\mathscr{A}\to\mathscr{V}$ is accessible when $\mathscr{A}$ is small.)

If $\mathscr{F}$ is any set of accessible $\mathscr{V}$-functors which have codomain $\mathscr{V}$, *$\mathscr{F}$-colimits* are colimits weighted (or indexed) by elements of $\mathscr{F}$, and an $\mathscr{F}$-cocomplete $\mathscr{V}$-category is a $\mathscr{V}$-category admitting $\mathscr{F}$-colimits. If $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are $\mathscr{F}$-cocomplete, then a functor from $\mathscr{A}$ to $\mathscr{B}$ is called *$\mathscr{F}$-cocontinuous* if its preserves all $\mathscr{F}$-colimits, and a functor $F:\mathscr{A}\otimes\mathscr{B}\to\mathscr{C}$ is called *separately $\mathscr{F}$-cocontinuous* if $F(A,-):\mathscr{B}\to\mathscr{C}$ and $\mathscr{F}(-,B):\mathscr{A}\to\mathscr{C}$ are $\mathscr{F}$-concontinuous for all $A\in\mathscr{A}$ and $B\in\mathscr{B}$. We write $\mathscr{F}$-Coc$[\mathscr{A},\mathscr{B}]$ for the full subcategory of $[\mathscr{A},\mathscr{B}]$ determined by the $\mathscr{F}$-cocontinuous functors, and $\mathscr{F}$-Coc$(\mathscr{A},\mathscr{B})$ for its underlying ordinary category. Similarly $S\mathscr{F}$-$Coc[\mathscr{A}\otimes\mathscr{B},\mathscr{C}]$ will denote the full subcategory of $[\mathscr{A}\otimes\mathscr{B},\mathscr{C}]$ determined by the separately $\mathscr{F}$-cocontinuous functors and $S\mathscr{F}$-$Coc(\mathscr{A}\otimes\mathscr{B},\mathscr{C})$ will denote its underlying ordinary category. All of this notation follows that of Kelly [9].

For a $\mathscr{V}$-category $\mathscr{A}$ let $\mathscr{F}\mathscr{A}$ denote the closure in $[\mathscr{A}^{\mathrm{op}},\mathscr{V}]$ of $\mathscr{A}$ under $\mathscr{F}$-colimits and let $y=y_{\mathscr{A}}:\mathscr{A}\to\mathscr{F}\mathscr{A}$ denote the Yoneda embedding seen as landing in $\mathscr{F}\mathscr{A}$. Thus, letting $\mathscr{P}$ be the set of all accessible weights, we get by [9, Section 5.7] that $\mathscr{P}\mathscr{A}=Acc[\mathscr{A}^{\mathrm{op}},\mathscr{V}]$, the free cocompletion of $\mathscr{A}$. Kelly there gives a construction of $\mathscr{F}\mathscr{A}$ by transfinite induction and shows that $\mathscr{F}\mathscr{A}$ is the free $\mathscr{F}$-cocompletion of $\mathscr{A}$ in the sense that for any $\mathscr{F}$-cocomplete $\mathscr{B}$, composition with y is an equivalence $\mathscr{F}$-$Coc[\mathscr{F}\mathscr{A},\mathscr{B}]\simeq[\mathscr{A},\mathscr{B}]$, with inverse Lan_y (= left Kan extension along y). Furthermore, by [9, Theorem 5.56] (see also [2, Section 2]) we have:

Proposition 2.1. *If $\mathscr{B}$ is $\mathscr{F}$-cocomplete and if $F:\mathscr{F}\mathscr{A}\to\mathscr{B}$, then F has a right adjoint iff F is $\mathscr{F}$-cocontinuous and $\mathscr{B}(F\cdot y_{\mathscr{A}}-,B)\in\mathscr{F}\mathscr{A}$ for all $B\in\mathscr{B}$.*

An accessible functor $F:\mathscr{K}^{\mathrm{op}}\to\mathscr{V}$ is called an *absolute weight* if all colimits weighted by F are absolute (i.e. preserved by any functor). Let $\mathscr{Q}$ be the set of all absolute weights. For small $\mathscr{A}$, Street shows in [14] that $\mathscr{Q}\mathscr{A}$ is just the Cauchy completion of $\mathscr{A}$ (defined in [9, Section 5.5] as the full subcategory of $\mathscr{P}\mathscr{A}$ determined by the small projectives). In this case, since every functor preserves all absolute colimits, $\mathscr{Q}\text{-}Coc[\mathscr{A},\mathscr{C}]=[\mathscr{A},\mathscr{C}]$ and $S\mathscr{Q}\text{-}Coc[\mathscr{A}\otimes\mathscr{B},\mathscr{C}]=[\mathscr{A}\otimes\mathscr{B},\mathscr{C}]$ for any Cauchy-complete $\mathscr{C}$.

We let $\mathscr{A}=(\mathscr{A},\circ,K)$ denote a monoidal $\mathscr{V}$-category where $\circ:\mathscr{A}\otimes\mathscr{A}\to\mathscr{A}$ is the tensor product of $\mathscr{A}$ and where K is the unit. As in [7], $\Phi=(\phi,\tilde{\phi},\phi^0):\mathscr{A}\to\mathscr{A}'$ denotes a monoidal functor where $\phi:\mathscr{A}\to\mathscr{A}'$ is a $\mathscr{V}$-functor, $\tilde{\phi}:\phi A\circ'\phi B\to\phi(A\circ B)$ is a natural transformation and $\phi^0:K'\to\phi K$ is an arrow in $\mathscr{A}'$ satisfying the usual coherence conditions. Recall that Φ is *strong* if $\tilde{\phi}$ and ϕ^0 are isomorphisms. Also recall that a monoidal natural transformation is just a natural transformation subject to two coherence conditions. We denote the resulting 2-category of (strong) monoidal categories by $[Str]Mon$. We will sometimes combine these prefixes with the prefix $\mathscr{F}\text{-}Coc$ so that, for instance, $StrMon\mathscr{F}\text{-}Coc(\mathscr{A},\mathscr{C})$ will denote the (ordinary) category of strong monoidal $\mathscr{F}$-cocontinuous functors from $\mathscr{A}$ to $\mathscr{C}$ and monoidal natural transformations between them. The results here will be proved for monoidal $\mathscr{V}$-categories, but the corresponding results will also hold for symmetric monoidal ones, with essentially unchanged proofs.

We recall [7, Proposition 2.2] due to Kelly in [8]:

Proposition 2.2. *Let $\Phi=(\phi,\tilde{\phi},\phi^0)$ be a monoidal functor. In order that Φ be a left adjoint in Mon, it is necessary and sufficient that ϕ be a left adjoint in $\mathscr{V}$-Cat and that Φ be strong. In fact, if $\eta,\varepsilon:\phi\dashv\psi$ is an adjunction in $\mathscr{V}$-Cat and Φ is strong, there is a unique enrichment of ψ to a monoidal Ψ (not in general strong) that renders η and ε monoidal; so that $\eta,\varepsilon:\Phi\dashv\Psi$ in Mon. Hence the monoidal Φ is an equivalence in Mon if and only if Φ is strong and ϕ is an equivalence in $\mathscr{V}$-Cat. The same results hold in the symmetric monoidal case.*

3. The free monoidal $\mathscr{F}$-cocompletion

Suppose that $\mathscr{A}$ and $\mathscr{B}$ are $\mathscr{V}$-categories and that $\mathscr{C}$ is an $\mathscr{F}$-cocomplete $\mathscr{V}$-category for some set $\mathscr{F}$ of weights. Letting $R:S\mathscr{F}\text{-}Coc(\mathscr{F}\mathscr{A}\otimes\mathscr{F}\mathscr{B},\mathscr{C})\to[\mathscr{A}\otimes\mathscr{B},\mathscr{C}]_0$ be the functor derived by composition with $y\otimes y:\mathscr{A}\otimes\mathscr{B}\to\mathscr{F}\mathscr{A}\otimes\mathscr{F}\mathscr{B}$, we have a generalisation of [7, Proposition 3.1]:

Proposition 3.1. *The functor R is an equivalence*

$$S\mathscr{F}\text{-}Coc(\mathscr{F}\mathscr{A}\otimes\mathscr{F}\mathscr{B},\mathscr{C})\simeq[\mathscr{A}\otimes\mathscr{B},\mathscr{C}]_0.$$

Proof. The inverse of R is the underlying functor of $L:[\mathscr{A}\otimes\mathscr{B},\mathscr{C}]\simeq[\mathscr{A},[\mathscr{B},\mathscr{C}]]\simeq\mathscr{F}\text{-}Coc[\mathscr{F}\mathscr{A},\mathscr{F}\text{-}Coc[\mathscr{F}\mathscr{B},\mathscr{C}]]\simeq S\mathscr{F}\text{-}Coc[\mathscr{F}\mathscr{A}\otimes\mathscr{F}\mathscr{B},\mathscr{C}]$ which takes a functor $T:\mathscr{A}\otimes\mathscr{B}\to\mathscr{C}$ to its left Kan extension along $y\otimes y$. $\square$

Now let $\mathscr{A} = (\mathscr{A}, \circ, K)$ be monoidal. From [4] there is a monoidal structure on $\mathscr{P}\mathscr{A}$ with unit $J = \mathscr{A}(-, K)$, and tensor product $*$ defined by the convolution formula

$$(f*g)A = \int^{B, C \in \mathscr{A}} fB \otimes gC \otimes \mathscr{A}(A, B \circ C)$$

$$\cong \operatorname{colim}(f-, \operatorname{colim}(g?, \mathscr{A}(A, - \circ ?)))$$

for $A \in \mathscr{A}$ and $f, g \in \mathscr{P}\mathscr{A}$. Although this result is only stated for small $\mathscr{A}$ in [4], it is clearly valid even when $\mathscr{A}$ is not small (though the monoidal $\mathscr{P}\mathscr{A}$ may not be closed in that case). From [1], we know that $\mathscr{F}\mathscr{A}$ is closed in $\mathscr{P}\mathscr{A}$ under f-weighted colimits whenever $f : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ is an object of $\mathscr{F}\mathscr{K}$. Thus, $*$ restricts to a tensor product of $\mathscr{F}\mathscr{A}$ which we will denote by $* : \mathscr{F}\mathscr{A} \otimes \mathscr{F}\mathscr{A} \to \mathscr{F}\mathscr{A}$. The unit $J = \mathscr{A}(-, K)$ is in $\mathscr{F}\mathscr{A}$ as this always contains the representables. The associativity and unit isomorphisms of $\mathscr{P}\mathscr{A}$ and the strong monoidal enrichment $y : \mathscr{A} \to \mathscr{P}\mathscr{A}$ of the Yoneda embedding from [7] all restrict to $\mathscr{F}\mathscr{A}$. Thus we have the following generalisation of Day's result [4], as re-formulated in part of [7, Proposition 4.1]:

Proposition 3.2. *If $\mathscr{A} = (\mathscr{A}, \circ, K)$ is (symmetric) monoidal, then $\mathscr{F}\mathscr{A}$ has a (symmetric) monoidal structure $(\mathscr{F}\mathscr{A}, *, J)$ and there is a strong monoidal inclusion $y = y_{\mathscr{A}} : \mathscr{A} \to \mathscr{F}\mathscr{A}$.*

Of course, we do not in general have (as in the case when $\mathscr{F} = \mathscr{P}$ and $\mathscr{A}$ is small) that $\mathscr{F}\mathscr{A}$ is biclosed even if $\mathscr{A}$ is. For instance, the countable colimit closure of the ordinary Cartesian closed category **1** is not Cartesian closed.

A monoidal $\mathscr{C} = (\mathscr{C}, *', J')$ is called *monoidally $\mathscr{F}$-cocomplete* if $\mathscr{C}$ is $\mathscr{F}$-cocomplete and $*' : \mathscr{C} \otimes \mathscr{C} \to \mathscr{C}$ is separately $\mathscr{F}$-cocontinuous. The proof of [7, Theorem 5.1] now generalises (using Proposition 2.1 above) easily to give

Theorem 3.3. *For a monoidal $\mathscr{A}$ and a monoidally $\mathscr{F}$-cocomplete $\mathscr{C}$, the functor $R : Mon\mathscr{F}\text{-}Coc(\mathscr{F}\mathscr{A}, \mathscr{C}) \to Mon(\mathscr{A}, \mathscr{C})$ given by composition with y is an equivalence of categories which restricts to an equivalence $StrMon\mathscr{F}\text{-}Coc(\mathscr{F}\mathscr{A}, \mathscr{C}) \simeq StrMon(\mathscr{A}, \mathscr{C})$. Moreover, the monoidal $F : \mathscr{F}\mathscr{A} \to \mathscr{C}$ has a right adjoint in Mon iff $F \in StrMon\mathscr{F}\text{-}Coc(\mathscr{F}\mathscr{A}, \mathscr{C})$ and $\mathscr{C}(F \circ y_{\mathscr{A}}-, C) \in \mathscr{F}\mathscr{A}$ for all $C \in \mathscr{C}$. The corresponding results are true in the symmetric monoidal case.*

4. Monoidal Morita equivalence

Of special interest is the case $\mathscr{F} = \mathscr{Q}$ where $\mathscr{Q}\mathscr{A}$ is the Cauchy completion of $\mathscr{A}$. We shall use $q = q_{\mathscr{A}} : \mathscr{A} \to \mathscr{Q}\mathscr{A}$ to denote the Yoneda embedding in this case. A monoidal $\mathscr{V}$-category $\mathscr{C}$ is called *monoidally Cauchy complete* if it is monoidally $\mathscr{Q}$-cocomplete. Since any tensor product is separately $\mathscr{Q}$-cocontinuous, $\mathscr{C}$ is monoidally Cauchy complete iff it is monoidal and Cauchy complete as a $\mathscr{V}$-category. The results of the previous section give:

Corollary 4.1. *For any monoidal $\mathscr{A}$, Day's monoidal structure on $\mathscr{P}\mathscr{A}$ restricts to $\mathscr{Q}\mathscr{A}$ so that there is a strong monoidal enrichment $\mathbf{q} : \mathscr{A} \to \mathscr{Q}\mathscr{A}$ of the functor q. If $\mathscr{C}$ is any monoidally Cauchy complete $\mathscr{V}$-category, the functor $R : Mon(\mathscr{Q}\mathscr{A}, \mathscr{C}) \to Mon(\mathscr{A}, \mathscr{C})$ given by composition with $\mathbf{q}$ is an equivalence of categories which restricts to an equivalence $StrMon(\mathscr{Q}\mathscr{A}, \mathscr{C}) \simeq StrMon(\mathscr{A}, \mathscr{C})$. We call $\mathscr{Q}\mathscr{A}$ the monoidal Cauchy completion of $\mathscr{A}$.*

By [1, Section 3] or Section 2 above, the cocontinuous functor L_q, unique to within isomorphism, for which we have

$$\begin{tikzcd} \mathscr{A} \arrow[r, "y_{\mathscr{A}}"] \arrow[d, "q"'] \arrow[dr, phantom, "\cong"] & \mathscr{P}\mathscr{A} \arrow[d, "L_q"] \\ \mathscr{Q}\mathscr{A} \arrow[r, "y_{\mathscr{Q}\mathscr{A}}"'] & \mathscr{P}\mathscr{Q}\mathscr{A} \end{tikzcd}$$

is in fact the left Kan extension along $y_{\mathscr{A}}$ of $y_{\mathscr{Q}\mathscr{A}} \cdot q$. Indeed, by [1, Section 3] $L_q F = Lan_{q^{\mathrm{op}}} F$. Because $\mathscr{A}$ is small, L_q has by [9, Theorem 4.51] a right adjoint which is easily seen to be $\mathscr{P}q$ given by composition with q^{op}. From Lindner's result [11, Proposition 3.4] or [9, Theorem 5.27] we know that this adjunction $L_q \dashv \mathscr{P}q$ is an adjoint equivalence. If $\mathscr{A}$ is monoidal this equivalence enriches to a monoidal equivalence.

Proposition 4.2. *Let $\mathscr{A}$ be small monoidal and let $\mathscr{P}\mathscr{A}$, $\mathscr{Q}\mathscr{A}$ and $\mathscr{P}\mathscr{Q}\mathscr{A}$ have the monoidal structures derived as above from that of $\mathscr{A}$. Then there are monoidal enrichments $\mathbf{L}_q : \mathscr{P}\mathscr{A} \to \mathscr{P}\mathscr{Q}\mathscr{A}$ and $\mathscr{P}\mathbf{q} : \mathscr{P}\mathscr{Q}\mathscr{A} \to \mathscr{P}\mathscr{A}$ of L_q and $\mathscr{P}_q$ respectively, such that $\mathbf{L}_q \dashv \mathscr{P}\mathbf{q}$ is an adjoint equivalence $\mathscr{P}\mathscr{A} \simeq \mathscr{P}\mathscr{Q}\mathscr{A}$ in Mon.*

Proof. Since $\mathbf{y}_{\mathscr{Q}\mathscr{A}}$ and $\mathbf{q}$ are strong monoidal, so is their composite $\mathbf{y}_{\mathscr{Q}\mathscr{A}} \cdot \mathbf{q}$. Thus by Theorem 3.3 there is a unique (up to isomorphism) strong monoidal cocontinuous functor $\mathbf{L}_q : \mathscr{P}\mathscr{A} \to \mathscr{P}\mathscr{Q}\mathscr{A}$ such that

$$\begin{tikzcd} \mathscr{A} \arrow[r, "y_{\mathscr{A}}"] \arrow[d, "q"'] \arrow[dr, phantom, "\cong"] & \mathscr{P}\mathscr{A} \arrow[d, "L_q"] \\ \mathscr{Q}\mathscr{A} \arrow[r, "y_{\mathscr{Q}\mathscr{A}}"'] & \mathscr{P}\mathscr{Q}\mathscr{A} \end{tikzcd}$$

where the isomorphism is monoidal. Clearly $\mathbf{L}_q$ is a monoidal enrichment of (some choice for) L_q.

By Proposition 2.2 there is a unique monoidal enrichment $\mathscr{P}\mathbf{q}$ of $\mathscr{P}q$ giving a monoidal adjoint equivalence $\mathbf{L}_q \dashv \mathscr{P}\mathbf{q}$. $\square$

We know from Lindner's [11, Proposition 3.9] that the opposite of any Cauchy complete $\mathscr{V}$-category is Cauchy complete and that if $\mathscr{A}$ is small the left Kan extension of $q^{\mathrm{op}}_{\mathscr{A}^{\mathrm{op}}} : \mathscr{A} \to \mathscr{Q}(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}$ along $q_{\mathscr{A}}$ gives an equivalence of $\mathscr{V}$-categories $\widehat{q^{\mathrm{op}}_{\mathscr{A}^{\mathrm{op}}}} : \mathscr{Q}\mathscr{A} \to \mathscr{Q}(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}$. Thus, since q preserves limits, it also preserves colimits, unlike most of the embeddings $y_{\mathscr{A}} : \mathscr{A} \to \mathscr{F}\mathscr{A}$. Again, this equivalence $\mathscr{Q}\mathscr{A} \simeq \mathscr{Q}(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}$ enriches to a monoidal equivalence if $\mathscr{A}$ is monoidal.

Proposition 4.3. *If $\mathscr{A}$ is small monoidal and $\mathscr{Q}\mathscr{A}$ is given the monoidal structure derived from that of $\mathscr{A}$, then $\mathscr{Q}\mathscr{A} \simeq \mathscr{Q}(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}$ monoidally.*

Proof. Since $q_{\mathscr{A}^{\mathrm{op}}}$ is a strong monoidal functor, there is a strong monoidal $q^{\mathrm{op}}_{\mathscr{A}^{\mathrm{op}}} : \mathscr{A} \to \mathscr{Q}(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}$ obtained by taking the inverse of the isomorphism $\mathscr{A}(\mathscr{Q}, A) * \mathscr{A}(-, B) \cong \mathscr{A}(-, A \circ B)$. Since $\mathscr{Q}(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}$ is Cauchy complete, it is monoidally Cauchy complete. Hence by Corollary 4.1 there is a unique (up to isomorphism) strong monoidal functor $\widehat{q^{\mathrm{op}}_{\mathscr{A}^{\mathrm{op}}}}$ such that

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "q_{\mathscr{A}}"] \arrow[dr, "q^{\mathrm{op}}_{\mathscr{A}^{\mathrm{op}}}"'] & \cong & \mathscr{Q}\mathscr{A} \arrow[dl, "\widehat{q^{\mathrm{op}}_{\mathscr{A}^{\mathrm{op}}}}"] \\
& \mathscr{Q}(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}. &
\end{tikzcd}$$

By the Cauchy completion property $\widehat{q^{\mathrm{op}}_{\mathscr{A}^{\mathrm{op}}}}$ is an enrichment of (some choice for) $\widehat{q^{\mathrm{op}}_{\mathscr{A}^{\mathrm{op}}}}$ which is an equivalence. Hence by Proposition 2.2, $\widehat{q^{\mathrm{op}}_{\mathscr{A}^{\mathrm{op}}}}$ which is an equivalence. Hence by Proposition 2.2, $q^{\mathrm{op}}_{\mathscr{A}^{\mathrm{op}}}$ is a monoidal equivalence. $\square$

The last two propositions give us the following, which was proved in the non-monoidal context in [11, Corollary 3.7].

Theorem 4.4. *If $\mathscr{A}$ and $\mathscr{B}$ are small monoidal $\mathscr{V}$-categories, then the following are equivalent* (*where all equivalences shown are monoidal*).

(i) $\mathscr{Q}\mathscr{A} \simeq \mathscr{Q}\mathscr{B}$.

(ii) $\mathscr{P}\mathscr{A} \simeq \mathscr{P}\mathscr{B}$.

(iii) $\mathscr{P}(\mathscr{A}^{\mathrm{op}}) = [\mathscr{A}, \mathscr{V}] \simeq [\mathscr{B}, \mathscr{V}] = \mathscr{P}(\mathscr{B}^{\mathrm{op}})$.

(iv) $\mathscr{Q}(\mathscr{A}^{\mathrm{op}}) \simeq \mathscr{Q}(\mathscr{B}^{\mathrm{op}})$.

Proof. (i) $\Rightarrow$ (ii). $\mathscr{P}\mathscr{A} \simeq \mathscr{P}\mathscr{Q}\mathscr{A}$ (by Proposition 4.2) $\simeq \mathscr{P}\mathscr{Q}\mathscr{B}$ (since $\mathscr{P} : MON^{\mathrm{coop}} \to MON$ is a 2-functor) $\simeq \mathscr{P}\mathscr{B}$.

(ii) $\Rightarrow$ (i). The equivalence of underlying $\mathscr{V}$-categories $\mathscr{P}\mathscr{A} \simeq \mathscr{P}\mathscr{B}$ restrict to an equivalence $\mathscr{Q}\mathscr{A} \simeq \mathscr{Q}\mathscr{B}$ which clearly enriches to a monoidal equivalence.

(iii) $\Leftrightarrow$ (iv). The dual of the above.

(i) $\Leftrightarrow$ (iv). Follows immediately from Proposition 4.3. $\square$

Of course the corresponding results for the symmetric monoidal case hold.

We end this section with the observation that the one-object case of *monoidal* Morita equivalence is (unlike the non-monoidal case) trivial.

Proposition 4.5. *If $\mathscr{A}$ and $\mathscr{B}$ are one-object monoidal $\mathscr{V}$-categories, then there is a monoidal equivalence $\mathscr{P}\mathscr{A} \simeq \mathscr{P}\mathscr{B}$ iff there is a monoidal isomorphism $\mathscr{A} \cong \mathscr{B}$.*

Proof. Let $*$ denote the one object of $\mathscr{A}$ or $\mathscr{B}$. Then if there is a monoidal equivalence $\Phi : \mathscr{P}\mathscr{A} \simeq \mathscr{P}\mathscr{B}$ we have by Yoneda that $\mathscr{A} = \mathscr{A}(*,*) \cong \mathscr{P}\mathscr{A}(\mathscr{A}(-,*), \mathscr{A}(-,*)) = \mathscr{P}\mathscr{A}(J,J) \cong \mathscr{P}\mathscr{B}(\phi J, \phi J) \cong \mathscr{P}\mathscr{B}(\mathscr{B}(-,*), \mathscr{B}(-,*)) \cong \mathscr{B}$, where these isomorphisms are all monoidal. □

Thus, for example, if R is a commutative ring with unit, ring multiplication is a monoidal tensor product on R giving rise (via the above convolution formula from [4]) to $\otimes_R$ on R-Mod. In this case any monoidal (or even unit-preserving) equivalence R-Mod $\simeq$ S-Mod must come from an isomorphism $R \cong S$. (This fact was pointed out to me by Dr. Martin Ward.)

5. Closed monoidal Cauchy completions

If $\mathscr{A}$ is a small closed monoidal $\mathscr{V}$-category, we can use the equivalence $(\mathscr{Q}\mathscr{A})^{\mathrm{op}} \simeq \mathscr{Q}(\mathscr{A}^{\mathrm{op}})$ and the equivalence $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{Q}\mathscr{A}] \simeq [\mathscr{Q}\mathscr{A}^{\mathrm{op}} \otimes \mathscr{Q}\mathscr{A}, \mathscr{Q}\mathscr{A}]$ (from Proposition 3.1) to see that $\mathscr{Q}\mathscr{A}$ is closed. More generally, however, $\mathscr{Q}\mathscr{A}$ may be closed even when $\mathscr{A}$ is only *near* closed in a sense we will make precise.

Definition. A $\mathscr{V}$-functor $G : \mathscr{B} \to \mathscr{A}$ is a *near right adjoint* to $F : A \to \mathscr{B}$ if there are natural transformations $\eta : 1 \to GF$ and $\varepsilon : FG \to 1$ such that

$$\begin{tikzcd} F \arrow[rr, "1"] \arrow[dr, "F\eta"'] & & F \\ & FGF \arrow[ur, "\varepsilon F"'] & \end{tikzcd}$$

or equivalently if $\mathscr{B}(F-, B)$ is a retract of $\mathscr{A}(-, GB)$ in $\mathscr{P}\mathscr{A}$, naturally in B.

If $F : \mathscr{A} \to \mathscr{B}$, we let $\mathscr{Q}F$ denote the unique functor such that

$$\begin{tikzcd} \mathscr{A} \arrow[r, "q_{\mathscr{A}}"] \arrow[d, "F"'] \arrow[dr, phantom, "\cong"] & \mathscr{Q}\mathscr{A} \arrow[d, "\mathscr{Q}F"] \\ \mathscr{B} \arrow[r, "q_{\mathscr{B}}"'] & \mathscr{Q}\mathscr{B} \end{tikzcd}$$

From the case $\mathscr{F} = \mathscr{Q}$ of Proposition 2.1, $\mathscr{Q}F$ has a right adjoint iff $\mathscr{B}(F-, B) \in \mathscr{Q}\mathscr{A}$ for all $B \in \mathscr{B}$. As $\mathscr{Q}\mathscr{A}$ always contains the retracts of the representables (and consists solely of them when $\mathscr{V} = \mathbf{Set}$) we get:

Proposition 5.1. *For any $\mathscr{A}$ we have the following chain of implications:*
(i) *$F : \mathscr{A} \to \mathscr{B}$ has a near right adjoint.*
$\Rightarrow$ (ii) *$\mathscr{Q}F : \mathscr{Q}\mathscr{A} \to \mathscr{Q}\mathscr{B}$ has a right adjoint.*
$\Rightarrow$ (iii) *$\mathscr{Q}F$ preserves any colimits which exist in $\mathscr{Q}\mathscr{A}$.*
$\Rightarrow$ (iv) *F preserves any colimits which exist in $\mathscr{A}$.*
If $\mathscr{V} = \mathbf{Set}$, then we also have (ii) $\Rightarrow$ (i), *though none of the implications can be reserved in general.*

Proof. For (iii) $\Rightarrow$ (iv) note that $q_{\mathscr{A}}$ preserves colimits and $q_{\mathscr{B}}$ is fully faithful. The rest follows from the above remarks. Alternatively, see Paré's result [12, Exercise 4 of Section IV.1] □

Definition. If $\mathscr{A}$ is monoidal, we say that $\mathscr{A}$ is *near closed* if each $- \circ A : \mathscr{A} \to \mathscr{A}$ has a near right adjoint.

For any small monoidal $\mathscr{A}$, the tensor $*$ of $\mathscr{P}\mathscr{A}$ derived from that of $\mathscr{A}$ is always separately cocontinuous and therefore $\mathscr{P}\mathscr{A}$ is biclosed. Let $[F, -]$ denote the right adjoint in $\mathscr{P}\mathscr{A}$ to $- * F$ for $F \in \mathscr{P}\mathscr{A}$.

Corollary 5.2. *If $\mathscr{A}$ is small monoidal, then we have the following chain of implications*:
(i) *$\mathscr{A}$ is near closed.*
$\Rightarrow$ (ii) *$\mathscr{Q}\mathscr{A}$ is closed (with the restriction of $[-, -]$ as the internal hom functor).*
$\Rightarrow$ (iii) *For all $A \in \mathscr{A}$, $- \circ A : \mathscr{A} \to \mathscr{A}$ preserves any colimits which exist in $\mathscr{A}$.*
If $\mathscr{V} = \mathbf{Set}$, then (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii). As $\mathscr{A}$ is near closed each $\mathscr{Q}(- \circ \mathscr{A}) \cong - * \mathscr{A}(-, A) : \mathscr{Q}\mathscr{A} \to \mathscr{Q}\mathscr{A}$ has a right adjoint $[\![\mathscr{A}(-, A), -]\!]$ by Proposition 5.1. Let $[\![-, -]\!] : \mathscr{Q}\mathscr{A}^{\mathrm{op}} \otimes \mathscr{Q}\mathscr{A} \to \mathscr{Q}\mathscr{A}$ be the unique functor (from Proposition 3.1) such that

$$\begin{tikzcd} \mathscr{A}^{\mathrm{op}} \otimes \mathscr{Q}\mathscr{A} \arrow[rr, "q^{\mathrm{op}} \otimes 1"] \arrow[dr, "{[\![q-, -]\!]}"'] & \arrow[d, phantom, "\cong"] & \mathscr{Q}\mathscr{A}^{\mathrm{op}} \otimes \mathscr{Q}\mathscr{A} \arrow[dl, "{[\![-, -]\!]}"] \\ & \mathscr{Q}\mathscr{A}. & \end{tikzcd}$$

To check that $[\![F, -]\!] : \mathscr{Q}\mathscr{A} \to \mathscr{Q}\mathscr{A}$ is a right adjoint to $- * F : \mathscr{Q}\mathscr{A} \to \mathscr{Q}\mathscr{A}$ we need to check the isomorphism $\mathscr{Q}\mathscr{A}(G * F, H) \cong \mathscr{Q}\mathscr{A}(G, [\![F, H]\!])$, but by Proposition 3.1 we need only check this for representable F and G for which we know it is true.

Finally, to see that $[\![-, -]\!]$ agrees with $[-, -]$ on $\mathscr{Q}\mathscr{A}$ note that for $A \in \mathscr{A}$, $[\![F, G]\!]A \cong \mathscr{P}\mathscr{A}(\mathscr{A}(-, A), [\![F, G]\!]) \cong \mathscr{P}\mathscr{A}(\mathscr{A}(-, A) * F, G) \cong \mathscr{P}\mathscr{A}(\mathscr{A}(-, A), [F, G]) \cong [F, G]A$ naturally in A.

(ii) $\Rightarrow$ (iii). Follows immediately from Proposition 5.1 as does (ii) $\Rightarrow$ (i) in the case $\mathscr{V} = \mathbf{Set}$. □

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SMALL CAUCHY COMPLETIONS*

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In this work it is shown that if the underlying category $\mathscr{V}_0$ of a symmetric closed monoidal category $\mathscr{V}$ is locally presentable, then the Cauchy completion of any small $\mathscr{V}$-category is small.

Introduction

It has been observed (e.g. by Kelly in [6]) that for many common monoidal categories $\mathscr{V}$ such as $\mathscr{V} = \mathbf{Set}$, $\mathbf{Cat}$, $\mathbb{R}^+$, or $\mathbf{AbGp}$, the Cauchy completion of a small $\mathscr{V}$-category is always small. Although Kelly gives a counterexample in [6] to show that this is not true for every closed, complete and cocomplete $\mathscr{V}$, it has been conjectured to be true for those $\mathscr{V}$ such that $\mathscr{V}_0$ is locally presentable. In some informal notes Kelly [5] proves this conjecture under the additional assumption that the unit I of $\mathscr{V}$ is projective for strong epis. Here we drop this assumption and prove that the Cauchy completion of a small $\mathscr{V}$-category is always small when the underlying category of $\mathscr{V}$ is locally presentable.

0. Notation

We use $\mathscr{V}$ (or $\mathscr{S}$) to denote a complete, cocomplete, symmetric monoidal closed category. If $\mathscr{A}$ is a small $\mathscr{V}$-category, then $\mathscr{P}\mathscr{A}$ will denote the $\mathscr{V}$-functor category $[\mathscr{A}^{\mathrm{op}}, \mathscr{V}]$ which, by [6, Theorem 4.51], is the free cocompletion of $\mathscr{A}$ under small colimits. We let $Y: \mathscr{A} \to \mathscr{P}\mathscr{A}$ denote the Yoneda embedding. If F and G are elements of $\mathscr{P}\mathscr{A}$, then G^F will abbreviate $\mathscr{P}\mathscr{A}(F, G) \in \mathscr{V}$. The identity of F is denoted by $j_F: I \to F^F$. We let K_G denote the canonical morphism: $\mathrm{colim}(G, Y^F) \to \mathrm{colim}(G, Y)^F \cong G^F$. If the underlying category of our base monoidal category is a

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† On 8 December 1988, shortly after submitting this article, and on the verge of the successful completion of his doctoral thesis, Scott Johnson met an untimely death at the age of 28. The parts of his thesis not contained in the present paper or reference [4] are being prepared for publication by Ross Street.

category of presheaves, then we shall denote the base monoidal category by $\mathcal{S}$. Throughout, $\mathcal{A}$ and $\mathcal{B}$ will denote *small* enriched categories.

1. Preliminaries

The equivalence of two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ in the bicategory $\mathcal{V}$-**Mod** (of modules between $\mathcal{V}$-categories as in [9]) is weaker than the equivalence of $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{V}$-**Cat**. This observation has led to the definition of the *Cauchy completion* $\mathcal{Q}\mathcal{A}$ of $\mathcal{A}$ such that $\mathcal{A} \simeq \mathcal{B}$ in $\mathcal{V}$-**Mod** if and only if $\mathcal{Q}\mathcal{A} \simeq \mathcal{Q}\mathcal{B}$ in $\mathcal{V}$-**Cat**. Lawvere [7] indicated a definition (made explicit in a more general context in [9]) of $\mathcal{Q}\mathcal{A}$ as the $\mathcal{V}$-category of modules $\mathcal{I} \to \mathcal{A}$ which possess a right adjoint in $\mathcal{V}$-**Mod**. Alternatively, $\mathcal{Q}\mathcal{A}$ is equivalent to the full subcategory of $\mathcal{P}\mathcal{A} = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ consisting of the *small projectives*: those F such that $\mathcal{P}\mathcal{A}(F,-) = (-)^F : \mathcal{P}\mathcal{A} \to \mathcal{V}$ preserves small colimits (see [6, Section 5.5] or [8]).

The following example from Kelly [6, Section 5.5] shows that $\mathcal{Q}\mathcal{A}$ need not be small when $\mathcal{A}$ is. Let $\mathbf{CL}_0$ be the category of complete lattices with sup-preserving functions and let $\otimes : \mathbf{CL}_0 \times \mathbf{CL}_0 \to \mathbf{CL}_0$ be such that the sup-preserving functions $A \otimes B \to C$ are the functions $A \times B \to C$ which are sup-preserving in each variable separately. This gives a monoidal category $\mathbf{CL}$ with the ordered set $\{0,1\}$ as unit.

Claim. *The Cauchy completion of a small* $\mathbf{CL}$*-category* $\mathcal{A}$ *is the full subcategory of* $[\mathcal{A}^{\mathrm{op}}, \mathbf{CL}]$ *consisting of those functors which are retracts of arbitrary (small) products (=coproducts) of representables. In particular,* $\mathcal{Q}\mathcal{A}$ *is not small unless* $\mathcal{A}$ *is equivalent to the one-object* $\mathbf{CL}$*-category with* $\mathcal{A}(*,*) = 0$.

Proof. Clearly, the coproduct of $\{A_i : i \in I\}$ in $\mathcal{V}_0$ is the same as the product $\prod_{i\in I} A_i$ with coprojection defined by

$$A_i \to \prod_{i \in I} A_i \to A_j,$$

$$a \mapsto \begin{cases} a & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we will denote this coproduct by $\bigoplus_{i\in I} A_i$. For any family $\{a_i : i \in I\}$ of objects of $\mathcal{A}$, any $F : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$, and any $G : \mathcal{K} \to \mathcal{P}\mathcal{A}$ with I and $\mathcal{K}$ small:

$$\mathcal{P}\mathcal{A}\Big(\bigoplus_{i\in I} \mathcal{A}(-,a_i), \operatorname{colim}(F,G)\Big) \cong \bigoplus_{i \in I} \operatorname{colim}(F, Ga_i) \cong \operatorname{colim}\Big(F, \bigoplus_{i\in I} Ga_i\Big)$$

$$\cong \operatorname{colim}\Big(F, \mathcal{P}\mathcal{A}\Big(\bigoplus_{i\in I} \mathcal{A}(-,a_i), G\Big)\Big).$$

Thus arbitrary products of representables, and hence their retracts (by [8, Corollary 3.6]) are small projective and so are in the Cauchy completion of $\mathcal{A}$.

Conversely, if F is small projective, the canonical morphism

$$K_F : \mathrm{colim}(F, Y^F) \to F^F$$

must be an isomorphism. In particular, K_F takes some element of its domain to 1_F. Since $\mathrm{colim}(F, Y^F)$ is a quotient of

$$\bigoplus_{a \in \mathscr{A}} Fa \otimes \mathscr{A}(-,a)^F \cong \bigoplus_{a \in \mathscr{A}} F^{\mathscr{A}(-,a)} \otimes \mathscr{A}(-,a)^F,$$

and since each $A \otimes B$ is itself a quotient of the complete lattice of all subsets of $A \times B$, there is a set I, and an I-indexed collection of pairs of morphisms $\{\langle x_i, y_i\rangle : i \in I\}$ with $x_i : F \to \mathscr{A}(-,a_i)$, and $y_i : \mathscr{A}(-,a_i) \to F$ such that $1_F = \sup_{i \in I}(y_i \circ x_i) : F \to F$. Thus F is a retract of $\bigoplus_{i \in I} \mathscr{A}(-,a_i)$. □

In the above proof, all that was needed for F to be small projective was that K_F map something onto the identity of F. A generalization of this idea to arbitrary $\mathscr{V}$ is given by Gouzou and Grunig [2, Theorem 1.1].

Proposition 1 (Gouzou and Grunig). *For any $\mathscr{V}$, if $F : \mathscr{A}^{\mathrm{op}} \to \mathscr{V}$ then F is small projective if and only if there is a morphism $\varphi : I \to \mathrm{colim}(F, Y^F)$ such that*

$$(*) \qquad \begin{tikzcd} I \arrow[rr, "\varphi"] \arrow[dr, "j_F"'] & & \mathrm{colim}(F, Y^F) \arrow[dl, "K_F"] \\ & F^F. & \end{tikzcd}$$

Proof. If F is small projective, we may take φ to be $K_F^{-1} \circ j_F$. So suppose φ satisfies $(*)$. To show that F is small projective, we need only show that $(-)^F$ preserves colimits of the form $\mathrm{colim}(G, Y)$ for $G : \mathscr{A}^{\mathrm{op}} \to \mathscr{V}$ since, for $G : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ and $H : \mathscr{K} \to \mathscr{P}\mathscr{A}$, $\mathrm{colim}(G, H) \cong \mathrm{colim}(\mathrm{colim}(G, H), Y) \cong \mathrm{colim}(G, \mathrm{colim}(H, Y))$. If $G : \mathscr{A}^{\mathrm{op}} \to \mathscr{V}$, then the composite

$$G^F \xrightarrow{\cong} G^F \otimes I \xrightarrow{1 \otimes \varphi} G^F \otimes \mathrm{colim}(F, Y^F) \xrightarrow{\text{can.}} \mathrm{colim}(G, Y^F),$$

is readily seen, using $(*)$, to be the inverse of the canonical $K_G : \mathrm{colim}(G, Y^F) \to G^F$. □

2. The presheaf case

Throughout this section, we assume that the underlying category of our base monoidal category is the category of presheaves $S^{\mathbb{C}^{\mathrm{op}}}$ for some small category $\mathbb{C}$ (where S is the category of sets). We denote our base category by $\mathscr{S} = (S^{\mathbb{C}^{\mathrm{op}}}, \otimes, I)$.

If X is a set, let $\|X\|$ denote its cardinality. If h is an $\mathrm{Obj}(\mathbb{C})$-graded set, let $\|h\| = \sum_{c \in \mathbb{C}} \|h(c)\|$ and if h is the underlying object function of a functor $H : \mathbb{C}^{\mathrm{op}} \to S$, let

$\|H\| = \|h\|$. If $\mathscr{A}$ is a small $\mathscr{S}$-category and if f is an $\mathrm{Obj}(\mathbb{C}) \times \mathrm{Obj}(\mathscr{A})$-graded set, let $\|f\| = \sum_{a \in \mathscr{A}} \|fa\|$ and if f is the underlying object function of a functor $F: \mathscr{A}^{\mathrm{op}} \to \mathscr{S}$, let $\|F\| = \|f\|$. Finally, let $\|\mathbb{C}\|$ denote the cardinality of the set of arrows of $\mathbb{C}$.

We now fix a small $\mathscr{S}$-category $\mathscr{A}$ and choose a cardinal κ such that

(1) $\|\mathbb{C}\| \le \kappa$.

(2) $\|I\| \le \kappa$.

(3) $\|\mathbb{C}(-,c) \otimes \mathbb{C}(-,d)\| \le \kappa$ for all $c, d \in \mathbb{C}$.

(4) $\|\mathscr{A}(a,b)\| \le \kappa$ for all $a, b \in \mathscr{A}$ and $\|\mathrm{Obj}(\mathscr{A})\| \le \kappa$.

Since $\otimes : \mathscr{S} \times \mathscr{S} \to \mathscr{S}$ is separately cocontinuous, we have, for $F, G \in \mathscr{S}$;

$$F \otimes G = \int^{c,d \in \mathbb{C}} Fc \times Gd \times (\mathbb{C}(-,c) \otimes \mathbb{C}(-,d))$$

which together with (1) and (3) (and the construction of coends in $S^{\mathbb{C}^{\mathrm{op}}}$) gives:

(5) If $F, G \in \mathscr{S}$ with $\|F\| \le \kappa$ and $\|G\| \le \kappa$, then $\|F \otimes G\| \le \kappa$.

Lemma 2. *Suppose $F: \mathscr{A}^{\mathrm{op}} \to \mathscr{S}$ is a functor and f is a sub $\mathrm{Obj}(\mathbb{C}) \times \mathrm{Obj}(\mathscr{A})$-graded set of F with $\|f\| \le \kappa$. Then there is a subfunctor $[f]$ of F, containing f, such that $\|[f]\| \le \kappa$.*

Proof. Let $U: (\mathscr{S}^{\mathscr{A}^{\mathrm{op}}})_0 \rightarrowtail S^{\mathrm{Obj}(\mathbb{C}) \times \mathrm{Obj}(\mathscr{A})}$ be the ordinary functor taking an $\mathscr{S}$-functor $F: \mathscr{A}^{\mathrm{op}} \to \mathscr{S}$ to its underlying $\mathrm{Obj}(\mathbb{C}) \times \mathrm{Obj}(\mathscr{A})$-graded set. Then U is a (not necessarily fully faithful) inclusion with left adjoint $L: S^{\mathrm{Obj}(\mathbb{C}) \times \mathrm{Obj}(\mathscr{A})} \to (\mathscr{S}^{\mathscr{A}^{\mathrm{op}}})_0$ given by

$$Lf = \coprod_{\substack{c \in \mathbb{C} \\ a \in \mathscr{A}}} \mathscr{A}(-,a) \otimes \mathbb{C}(-,c) \times fac.$$

If f and F are as in the statement of the lemma, let $J: Lf \to F$ correspond under the adjunction $L \dashv U$ to $f \rightarrowtail UF$ and let $Lf \twoheadrightarrow [f] \rightarrowtail F$ be the epi-mono factorization (calculated pointwise) of J. The natural transformation $Lf \twoheadrightarrow [f]$ corresponds by adjunction to the inclusion $f \rightarrowtail U[f]$. Since $\|f\| \le \kappa$, (4) and (5) give $\|[f]\| \le \|(Lf)\| \le \kappa$. $\square$

Lemma 3. *Suppose $F: \mathscr{A}^{\mathrm{op}} \to \mathscr{S}$ is a functor, $\xi \in \mathscr{S}$ with $\|\xi\| \le \kappa$, and suppose $T: \xi \to \mathrm{colim}(F, Y^F)$. Then*

$$\begin{tikzcd} \xi \arrow[r, "T"] \arrow[d, "\upsilon"'] & \mathrm{colim}(F, Y^F) \arrow[d, "K_F"] \\ G^F \arrow[r, tail, "i^F"] & F^F \end{tikzcd}$$

for some natural transformation υ and some inclusion $i: G \rightarrowtail F$ with $\|G\| \le \kappa$.

Proof. By [6, (3.70)],

$$\begin{aligned}\mathrm{colim}(F, Y^F) &\cong \int^{a\in\mathscr{A}} Fa \otimes \mathscr{A}(-,a)^F \\ &\cong \int^{a\in\mathscr{A}} \int^{d\in\mathbb{C}} Fad \times (\mathbb{C}(-,d)\otimes\mathscr{A}(-,a)^F).\end{aligned}$$

Thus there exist functions $\{t_c : c\in\mathbb{C}\}$ such that for all $c\in\mathbb{C}$,

$$\begin{tikzcd} & \xi c \arrow[dl, "t_c"'] \arrow[dr, "T_c"] & \\ \coprod\limits_{\substack{a\in\mathscr{A}\\ d\in\mathbb{C}}} Fad\times(\mathbb{C}(-,d)\otimes\mathscr{A}(-,a)^F)c \arrow[rr, two heads, "(e_F)c"] & & \mathrm{colim}(F, Y^F)c \end{tikzcd}$$

where e_F is the canonical natural transformation. Let $f=\{\pi_1(t_c(x))\in F : x\in\xi c$ for some $c\in\mathbb{C}\}$ and let $G=[f]:\mathscr{A}^{\mathrm{op}}\to\mathscr{S}$. By Lemma 2 and the assumption $\|\xi\|\leq\kappa$, we have $i: G\rightarrowtail F$ and $\|G\|\leq\kappa$. For all $c\in\mathbb{C}$,

$$\begin{tikzcd} \xi c \arrow[rr, "T_c"] \arrow[d, "t_c"'] & & \mathrm{colim}(F, Y^F)c \\ \coprod\limits_{\substack{a\in\mathscr{A}\\ d\in\mathbb{C}}} Gad\times(\mathbb{C}(-,d)\otimes\mathscr{A}(-,a)^F)c \arrow[rr, two heads, "(e_G)_c"] & & \mathrm{colim}(G, Y^F)c. \arrow[u, "\mathrm{colim}(i,1)_c"'] \end{tikzcd}$$

Now let υ_c be the composite $\xi c \xrightarrow{(e_G)_c\circ t_c} \mathrm{colim}(G, Y^F)c \xrightarrow{(K_G)_c} G^F c$. Then for all $c\in\mathbb{C}$,

$$\begin{tikzcd} \xi c \arrow[r, "T_c"] \arrow[d, "\upsilon_c"'] & \mathrm{colim}(F, Y^F)c \arrow[d, "(K_F)_c"] \\ G^F c \arrow[r, tail, "i_c^F"] & F^F c. \end{tikzcd}$$

Since each i_c^F is a monomorphism, the naturality of υ follows from the naturality of $K_F T$. $\square$

In particular, suppose $F:\mathscr{A}^{\mathrm{op}}\to\mathscr{S}$ is small projective. Then by Proposition 1 and Lemma 3 (with $\xi=I$) there is a υ and an $i: G\rightarrowtail F$ with $\|G\|\leq\kappa$ such that

$$\begin{tikzcd} I \arrow[d, "\upsilon"'] \arrow[dr, "j_F"] & \\ G^F \arrow[r, tail, "i^F"] & F^F. \end{tikzcd}$$

This immediately gives a factorization

$$\begin{tikzcd} F \arrow[d, "\tilde{v}"'] \arrow[dr, "1_F"] & \\ G \arrow[r, tail, "i"] & F \end{tikzcd}$$

whence $F \cong G$. That is, for any small projective F, $\|F\| \leq \kappa$. Since $\mathscr{A}$ is small, there is only a small number of non-isomorphic such F and we have

Theorem 4. *If $\mathscr{A}$ is a small $\mathscr{S}$-category, then the Cauchy completion $\mathscr{Q}\mathscr{A}$ of $\mathscr{A}$ is small.* □

3. The locally-presentable case

We will now generalize Theorem 4 from $\mathscr{S}$ to those $\mathscr{V} = (\mathscr{V}_0, \otimes, I, [-,-])$ such that $\mathscr{V}_0$ is locally presentable. For ease of exposition we consider only the case where $\mathscr{V}_0$ is locally *finitely* presentable, the generalization to locally presentable being entirely straightforward. From Gabriel and Ulmer [1] there is, for such a $\mathscr{V}$, a small finitely-cocomplete category $\mathbb{C}$ such that $\mathscr{V}_0 \simeq \mathrm{Lex}(S^{\mathbb{C}^{\mathrm{op}}}) =$ the full subcategory of $S^{\mathbb{C}^{\mathrm{op}}}$ consisting of the left-exact (or finitely continuous) functors. We will therefore identify $\mathscr{V}_0$ with $\mathrm{Lex}(S^{\mathbb{C}^{\mathrm{op}}})$ for the rest of this section.

We let $y : \mathbb{C} \to \mathscr{V}_0$ be the Yoneda embedding seen as landing in $\mathscr{V}_0$ and we let $Y : \mathbb{C} \to S^{\mathbb{C}^{\mathrm{op}}}$ denote the usual Yoneda embedding. From [6, Section 5.10], $F : \mathbb{C}^{\mathrm{op}} \to S$ is left exact if and only if it is a filtered colimit of representables. Thus, *$\mathscr{V}_0$ is the free filtered-colimit completion of $\mathbb{C}$*. From [1], the inclusion $i : \mathscr{V}_0 \to S^{\mathbb{C}^{\mathrm{op}}}$ has a reflection $\sigma : S^{\mathbb{C}^{\mathrm{op}}} \to \mathscr{V}_0$.

Theorem 5. *Let $\mathscr{V}$, i and σ be as above. Then*

(i) *There is a unique (up to isomorphism) symmetric closed monoidal structure $\mathscr{S}\,(=(S^{\mathbb{C}^{\mathrm{op}}}, \otimes, I, [-,-]))$ on $S^{\mathbb{C}^{\mathrm{op}}}$ such that $i : \mathscr{V}_0 \to S^{\mathbb{C}^{\mathrm{op}}}$ has a strong monoidal enrichment $i : \mathscr{V} \to \mathscr{S}$.*

(ii) *The inclusion i preserves the internal homs of $\mathscr{V}$ so that we may view any $\mathscr{V}$-category (respectively $\mathscr{V}$-functor, respectively $\mathscr{V}$-natural transformation) as an $\mathscr{S}$-category (respectively $\mathscr{S}$-functor, respectively $\mathscr{S}$-natural transformation). Since i preserves limits, limits and colimits in a $\mathscr{V}$-category are the same as for the corresponding $\mathscr{S}$-category.*

(iii) *There is a strong monoidal enrichment $(\sigma, \sigma^0, \tilde{\sigma}) : \mathscr{S} \to \mathscr{V}$ of σ. This makes $\mathscr{V}$ a strong monoidal reflective subcategory of $\mathscr{S}$.*

(iv) *There is an isomorphism $[\sigma X, V] \cong [X, V]$ natural in $X \in S^{\mathbb{C}^{\mathrm{op}}}$ and $V \in \mathscr{V}_0$.*

(v) *The ordinary functor $\sigma : S^{\mathbb{C}^{\mathrm{op}}} \to \mathscr{V}_0$ is the underlying functor of an $\mathscr{S}$-functor $\sigma : \mathscr{S} \to \mathscr{V}$.*

Proof. (i) Since $\otimes : \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$ is separately cocontinuous and since $i : \mathcal{V}_0 \to S^{\mathbb{C}^{op}}$ preserves filtered colimits, the composite $i\otimes$ preserves filtered colimits separately in both variables. We let S-$\mathbf{Coc}[S^{\mathbb{C}^{op}} \times S^{\mathbb{C}^{op}}, S^{\mathbb{C}^{op}}]$ denote the full subcategory of $[S^{\mathbb{C}^{op}} \times S^{\mathbb{C}^{op}}, S^{\mathbb{C}^{op}}]$ consisting of the separately cocontinuous functors and we let S-$\mathbf{FilCoc}[\mathcal{V}_0 \times \mathcal{V}_0, S^{\mathbb{C}^{op}}]$ denote the full subcategory of $[\mathcal{V}_0 \times \mathcal{V}_0, S^{\mathbb{C}^{op}}]$ consisting of the functors which preserve filtered colimits separately in both variables. By a result of Im and Kelly [3], and its generalization in [4] to arbitrary classes of weights for colimits we get:

(a) $\qquad [\mathbb{C} \times \mathbb{C}, S^{\mathbb{C}^{op}}] \simeq S\text{-}\mathbf{Coc}[S^{\mathbb{C}^{op}} \times S^{\mathbb{C}^{op}}, S^{\mathbb{C}^{op}}]$,

(b) $\qquad [\mathbb{C} \times \mathbb{C}, S^{\mathbb{C}^{op}}] \simeq S\text{-}\mathbf{FilCoc}[\mathcal{V}_0 \times \mathcal{V}_0, S^{\mathbb{C}^{op}}]$.

These equivalences are given, from left to right by left Kan extension along $Y \times Y$ (in (a)) and $y \times y$ (in (b)) and from right to left by restriction along $Y \times Y$ (in (a)) and $y \times y$ (in (b)). Thus, if we first restrict $i\otimes \in S\text{-}\mathbf{FilCoc}[\mathcal{V}_0 \times \mathcal{V}_0, S^{\mathbb{C}^{op}}]$ along $y \times y : \mathbb{C} \times \mathbb{C} \to \mathcal{V}_0 \times \mathcal{V}_0$ and then take its left Kan extension along $Y \times Y : \mathbb{C} \times \mathbb{C} \to S^{\mathbb{C}^{op}} \times S^{\mathbb{C}^{op}}$ we get a separately cocontinuous tensor product on $S^{\mathbb{C}^{op}}$ (which we will also denote by $\otimes$). This tensor product is, by [6, Theorem 4.47], the left Kan extension of $i\otimes$ along $i \times i$ and restricts (to within isomorphism) to the tensor product of $\mathcal{V}$.

$$\begin{tikzcd} \mathcal{V}_0 \times \mathcal{V}_0 \arrow[r, hook, "i \times i"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\cong"] & S^{\mathbb{C}^{op}} \times S^{\mathbb{C}^{op}} \arrow[d, "\otimes = \mathrm{Lan}_{i \times i}\, i\otimes"] \\ \mathcal{V}_0 \arrow[r, hook, "i"] & S^{\mathbb{C}^{op}}. \end{tikzcd}$$

The equivalences (a) and (b), together with their one- and three-dimensional analogues allow us to induce the symmetry, unity and associativity isomorphisms of $\mathcal{V}$ to $S^{\mathbb{C}^{op}}$. Verification that these isomorphisms satisfy the coherence axioms for a monoidal category is an easy exercise which gives a monoidal structure $\mathcal{S} = (S^{\mathbb{C}^{op}}, \otimes, I)$ on $S^{\mathbb{C}^{op}}$. This structure is unique such that $\otimes$ is separately cocontinuous and such that i preserves $\otimes$ and I. Since the tensor product of $\mathcal{S}$ is separately cocontinuous, $\mathcal{S}$ is closed.

(ii) Let $\{-,-\}$ denote the internal-hom functor of $\mathcal{S}$. For $U, V, W \in \mathcal{V}$, $S^{\mathbb{C}^{op}}(W, \{U, V\}) \cong S^{\mathbb{C}^{op}}(W \otimes U, V) \cong \mathcal{V}_0(W \otimes U, V) \cong \mathcal{V}_0(W, [U, V]) \cong S^{\mathbb{C}^{op}}(W, [U, V])$. Since $\mathcal{V}_0$ is dense in $S^{\mathbb{C}^{op}}$, $[U, V] \cong \{U, V\}$, i.e. the strong monoidal inclusion $i : \mathcal{V}_0 \to S^{\mathbb{C}^{op}}$ preserves internal homs. Henceforth, we will let $[-,-]$ denote the internal-hom functor in $\mathcal{S}$ as well as in $\mathcal{V}$.

(iv) For $U, V \in \mathcal{V}$ and $X \in \mathcal{S}$, $S^{\mathbb{C}^{op}}(U, [X, V]) \cong S^{\mathbb{C}^{op}}(X, [U, V]) \cong S^{\mathbb{C}^{op}}(\sigma X, [U, V]) \cong S^{\mathbb{C}^{op}}(U, [\sigma X, V])$. Again, since $\mathcal{V}_0$ is dense in $S^{\mathbb{C}^{op}}$, $[X, V] \cong [\sigma X, V]$.

(iii) For $X, Y \in \mathcal{S}$ and $V \in \mathcal{V}$, $[\sigma(X \otimes Y), V] \cong [X \otimes Y, V] \cong [X, [Y, V]] \cong [\sigma X, [\sigma Y, V]] \cong [\sigma X \otimes \sigma Y, V]$, which gives a natural isomorphism $\tilde{\sigma}_{X,Y} : \sigma X \otimes \sigma Y \cong \sigma(X \otimes Y)$. The counit of the adjunction $\sigma \dashv i$ gives an isomorphism $\sigma^0 : \sigma I \cong I$ and $(\sigma, \sigma^0, \tilde{\sigma}) : \mathcal{S} \to \mathcal{V}$ is a strong monoidal enrichment of σ.

(v) It is easy to check that σ is the underlying functor of an $\mathcal{S}$-functor $\sigma : \mathcal{S} \to \mathcal{V}$ with

$$\begin{tikzcd}
{[X,Y]} \arrow[rr, "\sigma_{X,Y}"] \arrow[dr, "{[1,\eta_Y]}"'] & & {[\sigma X, \sigma Y]} \\
& {[X, \sigma Y]} \arrow[ur, "\cong"'] &
\end{tikzcd}$$

where $\eta : 1_{S^{\mathbb{C}^{op}}} \to i\sigma$ is the unit of the adjunction $\sigma \dashv i$. $\square$

Of course, limits in $\mathcal{V}_0$ are calculated as in $S^{\mathbb{C}^{op}}$, and any colimit in $\mathcal{V}_0$ is given by taking the reflection of the corresponding colimit in $S^{\mathbb{C}^{op}}$. We reserve the usual notation for colimits, (including coproducts and coends) for the colimits as calculated in $S^{\mathbb{C}^{op}}$. We will write $\sigma(\mathrm{colim}(F, G))$ to denote the F-weighted colimit of G as calculated in $\mathcal{V}_0$. From now on we will identify σV with V for $V \in \mathcal{V}_0$ since these are naturally isomorphic.

Letting $\mathrm{Fin}\,\mathbb{C}$ denote the finite-colimit closure of $\mathbb{C}$ in $S^{\mathbb{C}^{op}}$, we have, by [6, Proposition 5.41] that $S^{\mathbb{C}^{op}}$ is the free filtered-colimit completion of $\mathrm{Fin}\,\mathbb{C}$. Since $i : \mathcal{V}_0 \to S^{\mathbb{C}^{op}}$ preserves filtered colimits, $i\sigma : S^{\mathbb{C}^{op}} \to S^{\mathbb{C}^{op}}$ is the left Kan extension of its restriction to $\mathrm{Fin}\,\mathbb{C}$.

$$\begin{tikzcd}
\mathrm{Fin}\,\mathbb{C} \arrow[rr, hook] \arrow[dr] & \cong & S^{\mathbb{C}^{op}} \arrow[dl, "i\sigma"'] \arrow[d, "\sigma"] \\
& S^{\mathbb{C}^{op}} & \mathcal{V}_0 . \arrow[l, hook', "i"]
\end{tikzcd}$$

Thus $\sigma(G) = \int^{\xi \in \mathrm{Fin}\,\mathbb{C}} \sigma(\xi) \times S^{\mathbb{C}^{op}}(\xi, G)$. Since $y : \mathbb{C} \to \mathcal{V}_0$ preserves finite colimits, $\sigma(\xi) = \mathbb{C}(-, \mathrm{colim}(\xi, 1_{\mathbb{C}}))$ for $\xi \in \mathrm{Fin}\,\mathbb{C}$.

Theorem 6. *If $\mathcal{V}_0$ is locally (finitely) presentable and if $\mathcal{A}$ is a small $\mathcal{V}$-category, then the Cauchy completion $\mathcal{Q}\mathcal{A}$ of $\mathcal{A}$ is also small.*

Proof. Let $\mathcal{A}$ be a small $\mathcal{V}$-category and let κ be as in Section 2. If $F, G : \mathcal{A}^{op} \to \mathcal{S}$, G^F will denote $[\mathcal{A}^{op}, \mathcal{S}](F, G) = \int_{a \in \mathcal{A}} [Fa, Ga] \in \mathcal{S}$ which is isomorphic to $[\mathcal{A}^{op}; \mathcal{V}](F, G)$ if F and G land in $\mathcal{V}$ since limits and internal homs in $\mathcal{V}$ are preserved by the inclusion $i : \mathcal{V} \to \mathcal{S}$. Note that any $\mathcal{S}$-functor $F : \mathcal{A}^{op} \to \mathcal{V}$ is a $\mathcal{V}$-functor.

By Proposition 1, if $F : \mathcal{A}^{op} \to \mathcal{V}$ is small projective, then there is a morphism φ in $\mathcal{V}$ such that

$$\begin{tikzcd}
I \arrow[rr, "\varphi"] \arrow[dr, "j_F"'] & & \sigma(\mathrm{colim}(F, Y^F)) \cong \int^{\xi \in \mathrm{Fin}\,\mathbb{C}} \sigma(\xi) \times S^{\mathbb{C}^{op}}(\xi, \mathrm{colim}(F, Y^F)). \\
& F^F \arrow[ur, "\sigma(K_F)"'] &
\end{tikzcd}$$

For each $c \in \mathbb{C}$ we can assign to each $h \in Ic$ a triple $\xi_h \in \operatorname{Fin} \mathbb{C}$, $g_c(h) \in \sigma(\xi_h)c$ and $T_h : \xi_h \to \operatorname{colim}(F, Y^F)$ (representing the value $\varphi_c(h)$) such that there is a commutative diagram of *functions* (note that g may not be a natural transformation)

$$\begin{tikzcd} & Ic \arrow[dl, "g_c"'] \arrow[dr, "\varphi_c"] & \\ \coprod_{h \in Ic} \sigma(\xi_h)c \arrow[rr, "(\sigma(T_h)_c)_h"] & & \sigma(\operatorname{colim}(F, Y^F))c. \end{tikzcd}$$

Clearly $\|\xi\| \le \kappa$ for all $\xi \in \operatorname{Fin} \mathbb{C}$. By Lemma 3 there is, for each $c \in \mathbb{C}$ and $h \in Ic$, a functor $G_h : \mathscr{A}^{\mathrm{op}} \to \mathscr{S}$ with $\|G_h\| \le \kappa$, an inclusion $i_h : G_h \rightarrowtail F$ and a morphism υ_h such that

$$\begin{tikzcd} \xi_h \arrow[r, "T_h"] \arrow[d, "\upsilon_h"'] & \operatorname{colim}(F, Y^F) \arrow[d, "K_F"] \\ G_h^F \arrow[r, tail, "i_h^F"] & F^F. \end{tikzcd}$$

Hence, for each $c \in \mathbb{C}$

$$\begin{tikzcd} & Ic \arrow[dl, "g_c"'] \arrow[dr, "\varphi_c"] & \\ \coprod_{h \in Ic} \sigma(\xi_h)c \arrow[rr, "(\sigma(T_h)_c)_h"] \arrow[d, "\coprod_{h \in Ic} \sigma(\upsilon_h)_c"'] & & \sigma(\operatorname{colim}(F, Y^F))c \arrow[d, "\sigma(K_F)_c"] \\ \coprod_{h \in Ic} \sigma(G_h^F)c \arrow[rr, "(\sigma(i_h^F)_c)_h"] & & F^F c. \end{tikzcd}$$

Since I and $\mathbb{C}$ are bounded by κ there is a subfunctor $G_0 : \mathscr{A}^{\mathrm{op}} \to \mathscr{S}$ of F with $\|G_0\| \le \kappa$ which contains each G_h for $h \in Ic$, $c \in \mathbb{C}$. We have inclusions

$$\begin{tikzcd} G_h \arrow[rr, tail, "i_h"] \arrow[dr, tail, "i_h'"'] & & F \\ & G_0 \arrow[ur, tail, "i_0"'] & \end{tikzcd}$$

for $h \in Ic$, $c \in \mathbb{C}$. Since σ is an $\mathscr{S}$-functor, the composite $\sigma \circ G_0 : \mathscr{A}^{\mathrm{op}} \to \mathscr{V}$ is an $\mathscr{S}$-functor between two $\mathscr{V}$-categories and is therefore a $\mathscr{V}$-functor. In the ordinary category $[\mathscr{A}^{\mathrm{op}}, \mathscr{V}]_0$ let $\sigma \circ G_0 \xrightarrow{e} G \xrightarrow{m} F$ be a strong epi-mono factorization of $\sigma \circ i_0 : \sigma \circ G_0 \to F$. This exists since $\mathscr{A}(A, B) \otimes - : \mathscr{V}_0 \to \mathscr{V}_0$ preserves strong epimorphisms. Then

$$
\begin{tikzcd}
\sigma(G_h^F) \arrow[rr, "\sigma(i_h^F)"] \arrow[d, "\sigma(i_h'^F)"'] & & F^F \\
\sigma(G_0^F) \arrow[urr, "\sigma(i_0^F)"] \arrow[dr, "\sigma((\eta G_0)^F)"'] & & \\
& (\sigma \circ G_0)^F \arrow[uur, "(\sigma i_0)^F"] \arrow[r, "e^F"] & G^F. \arrow[uu, tail, "m^F"']
\end{tikzcd}
$$

Now let μ_c be the composite

$$
Ic \xrightarrow{g_c} \coprod_{h \in Ic} \sigma(\xi_h)c \xrightarrow{\coprod_{h \in Ic} \sigma(\upsilon_h)_c} \coprod_{h \in Ic} \sigma(G_h^F)c \longrightarrow G^F c
$$

where the last arrow is derived from the composite of the three lower arrows in the previous diagram. Then we have (since $i : \mathscr{V}_0 \to S^{\mathbb{C}^{\mathrm{op}}}$ preserves monomorphisms)

$$
\begin{tikzcd}
Ic \arrow[r, "\varphi_c"] \arrow[d, "\mu_c"'] & \sigma(\operatorname{colim}(F, Y^F))c \arrow[d, "\sigma(K_F)_c"] \\
G^F c \arrow[r, tail, "(m^F)_c"] & F^F c
\end{tikzcd}
$$

and the naturality of μ follows from that of $\sigma(K_F)\varphi$. Hence, as in the presheaf case, $F \cong G$. Since $\|G_0\| \leq \kappa$ and since, by [1], any object of $\mathscr{V}$ has only a small number of quotients, there can only be a small number of such F. $\square$

In [9], Street defines the Cauchy completion $\mathscr{Q}\mathscr{A}$ of $\mathscr{A}$ where $\mathscr{A}$ is a small category enriched over a bicategory $\mathscr{W}$ such that $\mathscr{W}$ and $\mathscr{W}^{\mathrm{op}}$ admit right liftings. Suppose $\mathscr{W}(U, V)$ is locally representable for all objects U and V of $\mathscr{W}$. Then the proof here can be modified to show that for small $\mathscr{A}$, the set of objects in $\mathscr{Q}\mathscr{A}$ over any given object U of $\mathscr{W}$ is small. In particular, if $\mathrm{Obj}(\mathscr{W})$ is small, then so is $\mathscr{Q}\mathscr{A}$.

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MODULATED BICATEGORIES

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ABSTRACT. The concept of regular category [1] has several 2-dimensional analogues depending upon which special arrows are chosen to mimic monics. Here, the choice of the conservative arrows, leads to our notion of faithfully conservative bicategory $\mathcal{K}$ in which two-sided discrete fibrations become the arrows of a bicategory $\mathcal{F} = \mathrm{DFib}(\mathcal{K})$. While the homcategories $\mathcal{F}(B, A)$ have finite limits, it is important to have conditions under which these finite "local" limits are preserved by composition (on either side) with arrows of $\mathcal{F}$. In other words, when are all fibrations in $\mathcal{K}$ flat? Novel axioms on $\mathcal{K}$ are provided for this, and we call a bicategory $\mathcal{H}$ modulated when $\mathcal{H}^{\mathrm{op}}$ is such a $\mathcal{K}$. Thus, we have **constructed** a proarrow equipment $(\,)_* : \mathcal{H} \longrightarrow \mathcal{M}$ (in the sense of [28]) with $\mathcal{M} = \mathcal{F}^{\mathrm{coop}}$. Moreover, $\mathcal{M}$ is locally finitely cocomplete and certain collages exist [23].

In the converse direction, if $\mathcal{M}$ is any locally countably cocomplete bicategory which admits finite collages [23], then the bicategory $\mathcal{M}^*$ of maps in $\mathcal{M}$ is modulated. {Recall from [26], page 266, that a 1-cell in a bicategory is called a *map* when it has a right adjoint.}

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

§0. Introduction

The characterisation of bicategories $\mathcal{W}$-Mod, whose objects are categories with homs enriched in the base bicategory $\mathcal{W}$ and whose arrows are modules, was achieved in [23]. The main requirements on a bicategory $\mathcal{M}$ that it should be biequivalent to $\mathcal{W}$-Mod are the existence of coproducts, Kleisli constructions and local colimits preserved by composition on either side with an arrow. We call such an $\mathcal{M}$ a *cosmos*. Any "Cauchy generating" set of objects of $\mathcal{M}$ would then form a suitable $\mathcal{W}$. Also see [5].

An arrow of $\mathcal{M}$ is called [26] a *map* when it has a right adjoint. An enriched category is called *Cauchy complete* when it admits all absolute (weighted) limits (or, equally, colimits) [13], [25]. The maps $f\colon A \longrightarrow B$ in $\mathcal{W}$-Mod can be identified with $\mathcal{W}$-functors $f\colon \mathcal{Q}A \longrightarrow \mathcal{Q}B$ between Cauchy completions. We write $\mathcal{M}^*$ for the subbicategory of $\mathcal{M}$ obtained by restricting the arrows to maps. A suitable notation for $(\mathcal{W}\text{-Mod})^*$ is thus $(\mathcal{W}\text{-Cat})_{cc}$.

The main motivation for the present paper was to attempt to characterise bicategories of the form $(\mathcal{W}\text{-Cat})_{cc}$. This means we looked for conditions on a bicategory $\mathcal{H}$ (involving limits, colimits, inherent factorisation systems, and the like) which would ensure that $\mathcal{H}$ was equivalent to $\mathcal{M}^*$ for a cosmos $\mathcal{M}$. Apart from its intrinsic interest for enriched category theory, this question is relevant to the work of Pitts [17] and to the theory of quantales [15] [4] [16].

We were also guided by the challenge of keeping external set theory to a minimum. So our development requires only first order ("elementary") concepts. The replacement of "finite" by "small" is easily achieved.

Because of results of Street [22] and Wood [29], there was no doubt about how to construct $\mathcal{M}$ from $\mathcal{H}$; the arrows had to be two-sided codiscrete cofibrations in $\mathcal{H}$. The problem was to find elementary conditions on $\mathcal{H}$ to ensure that $\mathcal{M}$ became a bicategory and had the desirable properties, especially local finite cocompleteness. Because the reader may be familiar with the construction of the bicategory of relations from a regular category, and since our situation resembles the dual of this, we have decided to present the work in terms of $\mathcal{K} = \mathcal{H}^{\mathrm{op}}$ rather than $\mathcal{H}$ itself. This has the advantage that fibrations are generally more familiar than cofibrations, and preservation of finite limits by composition ("tensoring") with a fibration is a familiar flatness condition.

In the first two sections, we review the required limits, special arrows and factorisation systems in a bicategory. An arrow is *conservative* when a 2-cell into its source can be declared invertible if its composite with our arrow is invertible. The dual of conservative is *liberal*, while *strong liberals* can be defined from conservatives similarly to the way that strong epimorphisms are derived from monomorphisms [9]. *Faithfully conservational bicategories* are defined analogously to regular categories in [6] beginning with conservative arrows in place of monomorphisms.The main result (Theorem 2.19) of these sections is that stability of strong liberals under pseudopullback implies each strong liberal is a coinverter of some 2-cell. In the third section, we interpret the concepts of the earlier sections in the 2-categories *Cat*, Cat_{cc}, *Lex*, *Rex*, and their duals. The calculus of discrete fibrations in a faithfully conservative bicategory $\mathcal{K}$ is developed in section 4; in particular, we construct the bicategory DFib$(\mathcal{K})$ whose objects are those of $\mathcal{K}$ and whose arrows are two-sided discrete fibrations.

In section 5, a bicategory $\mathcal{H}$ is defined to be modulated when $\mathcal{H}^{\mathrm{op}}$ is faithfully conservational and two extra axioms about strong conservatives in $\mathcal{H}$ are satisfied. The reader should keep in mind the example of $\mathcal{H} = (\mathcal{V}\text{-Cat})_{cc}$ where the strong conservatives are fully faithful functors between the Cauchy complete $\mathcal{V}$-enriched categories. We show that the bicategory $\mathcal{M} = \mathrm{CodCofib}(\mathcal{H}) = \mathrm{DFib}(\mathcal{H}^{\mathrm{op}})^{\mathrm{coop}}$ of codiscrete cofibrations in a modulated bicategory $\mathcal{H}$ is locally finitely cocomplete. This is deduced from the result that the "inclusion" $(\,)_*\colon \mathcal{H}\longrightarrow\mathcal{M}$ preserves tensors with certain finite categories. This inclusion also preserves, so that $\mathcal{M}$ has, finite coproducts. A principal result here is the construction of the collage of any arrow in $\mathcal{M}$. Some questions still remain. We are unable to show that $(\,)_*\colon \mathcal{H}\longrightarrow\mathcal{M}$ preserves pseudopushouts; perhaps another axiom is required. A related problem is that we are unable to construct general finite collages in $\mathcal{M}$. Then there is the question of whether each map in $\mathcal{M}$ is isomorphic to f_* for some arrow f of $\mathcal{H}$. These questions need to be resolved before our motivating problem can be fully answered.

Finally, in section 6, we show that, if $\mathcal{M}$ is any locally finitely cocomplete bicategory which admits finite collages and free monads on endoarrows, then $\mathcal{M}^*$ is modulated and the inclusion $\mathcal{M}^*\longrightarrow\mathcal{M}$ preserves finite colimits.

§1. Review of Limits in Bicategories.

1.1. The appropriate limits for bicategories were introduced in [22]. These limits differ from those appropriate for 2-categories [21] in that they are only unique up to equivalence, not up to isomorphism. Our purpose in this section is to recall the general definition and to give examples needed in the present paper. Since the general definition is a representable one, it suffices to give the examples in the bicategory *Cat* of categories, functors and natural transformations. In order to avoid the confusing "bi" prefix in terminology such as "bilimit" and "biproduct" (since these could also mean that they, at the same time, provide the dual notion), we have adopted the policy of using the 2-categorical name for the limit when it is understood that we are working bicategorically (even if we are working in a 2-category). However, one should be aware that some 2-categorical limits (pullbacks and equalizers for example) do not survive this passage to bicategories, and so have no rôle for them.

1.2. Suppose $J, S\colon \mathcal{A} \longrightarrow Cat$ are homomorphisms of bicategories. Write S^J for the category whose objects are strong transformations (=pseudo-natural transformations) $J \longrightarrow S$, and whose arrows are modifications [2]. Observe that, for all X, there is a canonical equivalence of categories

$$Cat(X, S^J) \xrightarrow{\sim} Cat(X, S)^J$$

1.3. Now suppose $J\colon \mathcal{A} \longrightarrow Cat$, $S\colon \mathcal{A} \longrightarrow \mathcal{K}$ are homomorphisms of bicategories. A *J-weighted limit* of S consists of an object S^J of $\mathcal{K}$ and a strong transformation $\lambda\colon J(-) \longrightarrow \mathcal{K}(S^J, S(-))$ such that, for all objects X of $\mathcal{K}$, the functor

$$\begin{aligned} \mathcal{K}(X, S^J) &\xrightarrow{\sim} \mathcal{K}(X, S)^J, \\ u &\longmapsto \mathcal{K}(u, S) \circ \lambda \end{aligned}$$

is an equivalence of categories. The representable nature of this general definition allows us to describe various examples by merely explaining them in *Cat*. We do not need to exhibit the weight homomorphism J in each case. The reader who has had no experience in this may refer to [22],[21] and [11].

A *J-weighted colimit* of a homomorphism $T\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{K}$ is a J-weighted limit of T regarded as a homomorphism $\mathcal{A} \longrightarrow \mathcal{K}^{\mathrm{op}}$. For each of our examples below there is a dual version for which the prefix "co" is used. We have no need here for examples which give new constructions when $\mathcal{K}$ is replaced by the weak dual $\mathcal{K}^{\mathrm{co}}$.

1.4. The *product* of two categories is defined in the obvious way by taking, as the set of arrows, the cartesian product of the sets of arrows. The *product* of objects A, B in $\mathcal{K}$ consists of an object $A \times B$, and arrows

$$A \xleftarrow{p_A} A \times B \xrightarrow{p_B} B$$

(called *projections*) in $\mathcal{K}$, such that the induced functor

$$\mathcal{K}(X, A \times B) \xrightarrow{\sim} \mathcal{K}(X, A) \times \mathcal{K}(X, B)$$

is an equivalence for all objects $X \in \mathcal{K}$. Of course, products of families of objects are now defined in the obvious way. In particular (the empty family case), a *terminal* *object* 1 for $\mathcal{K}$ is defined by the condition that each $\mathcal{K}(X, 1)$ is equivalent to the terminal category $\mathbb{1}$.

1.5. The comma category $f \downarrow g$ of two functors forming a cospan

$$A \xrightarrow{f} C \xleftarrow{g} B$$

is a familiar construction [14]. The objects are triples $(a, \gamma\colon fa \to gb, b)$ where a, b are objects of A, B and γ is an arrow of C. An arrow $(\alpha, \beta)\colon (a, \gamma, b) \longrightarrow (a', \gamma', b')$ consists of arrows $\alpha\colon a \longrightarrow a'$, $\beta\colon b \longrightarrow b'$ of A, B such that $\gamma' \circ f(\alpha) = g(\beta) \circ \gamma$. There is a diagram

(1)
$$\begin{tikzcd} f \downarrow g \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd}$$

where λ is the natural transformation whose component at the object (a, γ, b) is the arrow γ. The *comma object* of a cospan f, g as above, but now in our bicategory $\mathcal{K}$, consists of a diagram in $\mathcal{K}$ in the form of the last square, which induces an equivalence of categories

$$\mathcal{K}(X, f \downarrow g) \xrightarrow{\quad\simeq\quad} \mathcal{K}(X, f) \downarrow \mathcal{K}(X, g).$$

1.6. The *pseudopullback* of functors $f\colon A \longrightarrow C$, $g\colon B \longrightarrow C$ is the full subcategory $A_f \times_g B$ (or, less precisely $A \times_C B$) of $f \downarrow g$ consisting of those objects $(a, \gamma\colon fa \to gb, b)$ for which the arrow γ is invertible. This sits in a square like the one for the comma category (1), but this time the 2-cell in our square is invertible. The left arrow in this square is sometimes denoted by pr_1 and the top arrow by pr_2. It should be clear now how to define the *pseudopullback* of a cospan in $\mathcal{K}$. We also use the terminology that the arrow on the top (respectively, at the left) of a pseudopullback square is a *pseudopullback of the arrow* on the bottom (respectively, at the right).

1.7. Proposition. *Suppose that the right-hand square in the diagram*

$$\begin{tikzcd} K \arrow[r, "s"] \arrow[d, "r"'] \arrow[dr, phantom, "\cong"] & H \arrow[r, "q"] \arrow[d, "p"] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "g"] \\ D \arrow[r, "h"'] & A \arrow[r, "f"'] & C \end{tikzcd}$$

exhibits H as a comma object of f, g. the whole pasted diagram exhibits K as a comma object of fh, g if and only if the left-hand square is a pseudopullback of h, p. □

1.8. For categories $\mathbb{C}, \mathbb{D}$ we have the funtor category $\mathbb{D}^{\mathbb{C}}$. This leads to the *cotensor product* $A^{\mathbb{C}}$ of an object A of $\mathcal{K}$ with a category $\mathbb{C}$; which comes equipped with a functor $\mathbb{C} \longrightarrow \mathcal{K}(A^{\mathbb{C}}, A)$, inducing an equivalence of categories

$$\mathcal{K}(X, A^{\mathbb{C}}) \xrightarrow{\simeq} \mathcal{K}(X, A)^{\mathbb{C}}.$$

(Of course, the dual of cotensor is *tensor*, not cocotensor!)

1.9. The *pseudoequalizer* of two functors $f, g \colon A \longrightarrow B$ is the category C whose objects are pairs $(a, \beta \colon fa \xrightarrow{\simeq} ga)$ where a is an object of A and β is an invertible arrow of B, and whose arrows $\alpha \colon (a, \beta) \longrightarrow (a', \beta')$ are arrows $\alpha \colon a \longrightarrow a'$ of A such that $\beta' \circ f(\alpha) = g(\alpha) \circ \beta$. Also, we have the forgetful functor $u \colon C \longrightarrow A$ and a canonical invertible 2-cell $fu \Rightarrow gu$. This leads to the corresponding limit in $\mathcal{K}$: a *pseudoequalizer of a pair of arrows* $f, g \colon A \longrightarrow B$ is an arrow $u \colon C \longrightarrow A$ together with an invertible 2-cell $fu \Rightarrow gu$ which induces an equivalence between $\mathcal{K}(X, C)$ and the pseudoequalizer of the functors $\mathcal{K}(X, f)$, $\mathcal{K}(X, g)$ for all X.

1.10. The *inverter* of a natural transformation $\sigma \colon f \Rightarrow g \colon A \longrightarrow B$ is the full subcategory of A consisting of the objects a for which the component $\sigma_a \colon fa \longrightarrow ga$ is invertible. An *inverter* for a 2-cell $\sigma \colon f \Rightarrow g \colon A \longrightarrow B$ in $\mathcal{K}$ is thus an arrow $u \colon C \longrightarrow A$ which induces an equivalence between the category $\mathcal{K}(X, C)$ and the inverter of the natural transformation $\mathcal{K}(X, \sigma)$.

1.11. The *invertee* of a functor $f \colon A \longrightarrow B$ is the full subcategory C of the category $A^{\mathbf{2}}$ of arrows of A consisting of those arrows inverted by f. There is a canonical 2-cell $\lambda \colon u \Rightarrow v \colon C \longrightarrow A$ with $f\lambda$ invertible. The reader should now be able to define an *invertee* for an arrow in $\mathcal{K}$.

1.12. The *equifier* (resp. *equinverter*) of two natural transformations $\sigma, \tau \colon f \Rightarrow g \colon A \longrightarrow B$ is the full subcategory of A consisting of the objects a for which $\sigma_a = \tau_a$ (resp. $\sigma_a = \tau_a$ and σ_a is invertible). The bicategorical limit should be clear.

1.13. A bicategory $\mathcal{K}$ is said to be *finitely complete* when it admits all limits weighted by homomorphisms $J \colon \mathcal{A} \longrightarrow Cat$ such that $\mathcal{A}$ is a finite bicategory and each $J(A)$ is a finitely presentable category.

1.14. Theorem (Street [22] and [27]). *Every bicategory which admits a terminal object, pseudopullbacks, and cotensor products with the arrow category $\mathbf{2}$ is finitely complete.* □

1.15. A *pseudopushout* of a pair of arrows $j \colon X \longrightarrow Y$, $f \colon X \longrightarrow A$ in $\mathcal{K}$ is of course their pseudopullback in $\mathcal{K}^{\mathrm{op}}$. As a special example, we shall describe the pseudopushout P of j, f in Cat in the special case where X is a full subcategory of Y, closed under isomorphs (replete), and j is the inclusion. First the equality $\mathrm{obj}(P) = \mathrm{obj}(A) + (\mathrm{obj}(Y) - \mathrm{obj}(X))$ determines the objects of P. The category A is a full subcategory of P, let the inclusion be called $g \colon A \longrightarrow P$. The other homsets of P are given by coends

$$P(a, y) = \int^{x} A(a, fx) \times Y(jx, y), \quad P(y, a) = \int^{x} Y(y, jx) \times A(fx, a)$$

$$P(y, y') = \int^{x, x'} Y(y, jx) \times A(fx, fx') \times Y(jx', y'),$$

where $a \in A$, $y, y' \in Y$, $y, y' \notin X$. Composition in P is given in the obvious way using the compositions of A and Y. There is a functor $h\colon Y \longrightarrow P$ given by $hy = y$ for $y \notin X$, and $hx = fx$ for $x \in X$. So we have an actual equality $hj = gf$, not just an isomorphism. In fact (P, g, h) provide a pushout for j, f as well as a pseudopushout. This extends Proposition 6.17 of [22].

§2. Conservative and Liberal Arrows

2.1. An arrow $j\colon X \longrightarrow Y$ in a bicategory $\mathcal{K}$ is called *conservative* when, for all 2-cells $\sigma\colon u \Rightarrow v\colon K \longrightarrow X$, if $j\sigma$ is invertible then so is σ itself. That is, when, for all objects K, the functor

$$\mathcal{K}(K,X) \xrightarrow{\mathcal{K}(K,j)} \mathcal{K}(K,Y)$$

is conservative (= reflects isomorphisms). This condition holds precisely when the identity 2-cell of the identity arrow of X is an invertee (1.11) for j. Clearly, j is conservative in $\mathcal{K}$ iff it is conservative in $\mathcal{K}^{\mathrm{co}}$.

2.2. An arrow in $\mathcal{K}$ is called *liberal* when it is conservative in $\mathcal{K}^{\mathrm{op}}$.

2.3. Suppose $\mathcal{K}$ admits cotensoring with the arrow category $\mathbf{2}$. Then $j\colon X \longrightarrow Y$ is conservative iff the following square is a pseudopullback.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{diag}"] \arrow[d, "j"'] \arrow[dr, phantom, "\cong"] & X^{\mathbf{2}} \arrow[d, "j^{\mathbf{2}}"] \\
Y \arrow[r, "\mathrm{diag}"'] & Y^{\mathbf{2}}
\end{tikzcd}$$

2.4. Any arrow isomorphic to a conservative arrow is conservative. Any composite of conservative arrows is conservative. If kj is conservative then so is j. Conservative arrows are stable under pseudopullback.

2.5. Suppose $J\colon \mathcal{A} \longrightarrow Cat$, $S, T\colon \mathcal{A} \longrightarrow \mathcal{K}$ are homomorphisms of bicategories and suppose $\mathcal{K}$ admits the J-weighted limits S^J, T^J of S, T. If each component of a strong transformation $q\colon S \longrightarrow T$ is conservative then so is the induced arrow $q^J\colon S^J \longrightarrow T^J$.

2.6. An arrow $j\colon X \longrightarrow Y$ is *faithful* when, for all 2-cells $\sigma, \tau\colon u \Rightarrow v\colon K \longrightarrow X$, if $j\sigma = j\tau$ then $\sigma = \tau$. That is, when, for all objects K, the functor $\mathcal{K}(K,j)$ is faithful.

2.7. An arrow $j\colon X \longrightarrow Y$ is *pseudomonic* when it is faithful and, for all pairs of arrows $a, b\colon K \longrightarrow X$ and all invertible 2-cells $\zeta\colon ja \Rightarrow jb$, there exists a 2-cell $\eta\colon a \Rightarrow b$ such that $j\eta = \zeta$. (It follows that η is uniquely determined and invertible.) Note that j is pseudomonic iff the following square is a pseudopullback.

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\cong"] & X \arrow[d, "j"] \\
X \arrow[r, "j"'] & Y
\end{tikzcd}$$

Each pseudomonic is conservative. An arrow is *pseudoepic* in $\mathcal{K}$ when it is pseudomonic in $\mathcal{K}^{\mathrm{op}}$.

2.8. Suppose $\mathcal{K}$ admits cotensoring with the discrete category $2 \overset{\text{def}}{=} \mathbb{1} + \mathbb{1}$. Then $j\colon X \longrightarrow Y$ is pseudomonic iff the following square is a pseudopullback.

$$\begin{tikzcd} X \arrow[r, "\mathrm{diag}"] \arrow[d, "j"'] \arrow[dr, phantom, "\cong"] & X^2 \arrow[d, "j^2"] \\ Y \arrow[r, "\mathrm{diag}"'] & Y^2 \end{tikzcd}$$

2.9. All inverters and equinverters are pseudomonic.

2.10. A functor $f\colon A \longrightarrow B$ is an equivalence iff it is pseudomonic and the functor $f^2\colon A^2 \longrightarrow B^2$ is surjective up to isomorphism on objects.

2.11. An arrow $e\colon A \longrightarrow B$ in a bicategory $\mathcal{K}$ is called *strong liberal* (abbreviated to *slib*) when, for all conservative arrows $j\colon X \longrightarrow Y$, the following square is a pseudopullback of categories.

$$\begin{tikzcd} \mathcal{K}(B,X) \arrow[r, "\mathcal{K}(e,X)"] \arrow[d, "\mathcal{K}(B,j)"'] \arrow[dr, phantom, "\cong"] & \mathcal{K}(A,X) \arrow[d, "\mathcal{K}(A,j)"] \\ \mathcal{K}(B,Y) \arrow[r, "\mathcal{K}(e,Y)"'] & \mathcal{K}(A,Y) \end{tikzcd}$$

An arrow in $\mathcal{K}$ is *strong conservative* (abbreviated to *scon*) when it is slib in $\mathcal{K}^{\mathrm{op}}$.

2.12. Strong liberals are closed under composition. If eh is strong liberal and h is either pseudoepic or strong liberal then e is strong liberal. An arrow which is both strong liberal and conservative is an equivalence.

2.13. Every slib $e\colon A \longrightarrow B$ satisfies the following condition: *each square*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] \arrow[dr, phantom, "\cong"] & B \arrow[d, "v"] \\ X \arrow[r, "j"'] & Y, \end{tikzcd}$$

in which j is conservative, factorizes uniquely up to isomorphism as

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] \arrow[dr, phantom, "\cong", near start] & D \arrow[d, "v"] \arrow[dl, "w"] \arrow[dl, phantom, "\cong", shift left=3ex] \\ X \arrow[r, "j"'] & Y. \end{tikzcd}$$

When $\mathcal{K}$ admits cotensoring with the arrow category $\mathbf{2}$, this condition **implies** e is slib. (Apply the condition to both j and $j^{\mathbf{2}}$ to get the pseudopullback of (2.11).)

2.14. *If $\mathcal{K}$ admits all inverters then all strong liberals are liberal.* For, suppose $e\colon A \longrightarrow B$ is slib, and suppose θe invertible. Let j be an inverter for θ; so $e \cong ju$ for some u. By (2.12), since j is conservative, there exists w such that $u \cong we$ and $jw \cong 1$. The latter gives θ invertible (since θjw is). So e is liberal. □

2.15. *If $\mathcal{K}$ admits cotensoring with the discrete category 2 then strong liberal arrows are pseudoepic.* For all objects K, the diagonal $K \longrightarrow K^2$ is conservative. Substituting this for j in the pseudopullback of (2.11), we see from (2.8) that $\mathcal{K}(e, K)$ is pseudomonic. So e is pseudoepic.

2.16. Each left adjoint with invertible counit is the coinverter of the unit. Dually, each right adjoint with invertible unit is the coinverter of the counit. Our next result shows that this gives two interesting classes of strong liberals.

2.17. *Each coinverter is strong liberal.* This follows representably from the fact that, in the following diagram of ordinary categories

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\cong"] & Y \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow\theta"] \arrow[d, "v"] \arrow[dr, phantom, "\cong"] & Z \arrow[d, "w"] \\ X \arrow[r, "f'"'] & Y' \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow\theta'"] & Z' \end{tikzcd} ,$$

if f, f' are inverters of θ, θ' and w is conservative then the square u, f', v, f is a pseudopullback. □

2.18. Definition. A bicategory $\mathcal{K}$ will be called *conservational* when it satisfies the following conditions:

(i) it is finitely complete;
(ii) each pseudo pullback of a strong liberal is strong liberal;
(iii) cotensoring $(\ )^{\mathbf{2}}$ with the arrow category $\mathbf{2}$ preserves strong liberal arrows;
(iv) for each arrow f, there exists a conservative arrow j, a strong liberal arrow e, and an invertible 2-cell $f \Rightarrow je$.

2.19. Theorem. *In a conservational bicategory, every strong liberal arrow is a coinverter.*

Proof. For any arrow $e\colon A \longrightarrow B$, the invertee $\kappa_e\colon K_e \Rightarrow A$ of e corresponds to the pseudopullback $k_e\colon K_e \longrightarrow A^{\mathbf{2}}$ of the diagonal $B \longrightarrow B^{\mathbf{2}}$ along $e^{\mathbf{2}}\colon A^{\mathbf{2}} \longrightarrow B^{\mathbf{2}}$. We shall use this notation below.

Assume e is slib. We shall show that e is the coinverter of κ_e. In view of (2.15), (2.10) and the existence of $(\ )^{\mathbf{2}}$, it suffices to show that each $g\colon A \longrightarrow C$, with $g\kappa_e$ invertible, factors up to isomorphism through e. For this apply (2.18)(iv) to the arrow $(e, g)\colon A \longrightarrow B \times C$ to obtain a conservative $(u, v)\colon D \longrightarrow B \times C$ and a slib $s\colon A \longrightarrow D$ with $us \cong e$, $vs \cong g$. Since s, e are slib, so too is u (2.12). We have pseudopullbacks:

$$\begin{tikzcd} K_e \arrow[r, "r"] \arrow[d, "k_e"'] \arrow[dr, phantom, "\cong"] & K_u \arrow[r] \arrow[d, "k_u"'] \arrow[dr, phantom, "\cong"] & B \arrow[d, "\mathrm{diag}"] \\ A^{\mathbf{2}} \arrow[r, "s^{\mathbf{2}}"'] & D^{\mathbf{2}} \arrow[r, "u^{\mathbf{2}}"'] & B^{\mathbf{2}} \end{tikzcd}$$

Now $(u, v)\kappa_u r \cong (u, v)s\kappa_e \cong (e, g)\kappa_e \cong (e\kappa_e, g\kappa_e)$ is invertible. Since (u, v) is conservative, $\kappa_u r$ is invertible. By (2.18)(ii),(iii) and (2.14), r is liberal; so κ_u is invertible. So u is conservative. So u is an equivalence (2.10). So $g \cong vs \cong vwe$, where w is any inverse equivalence for u. □

2.20. Remark. Perhaps the reader will have noticed that in the above proof we needed only the weaker form of (2.18)(ii): a pseudopullback of a strong liberal is liberal.

2.21. *In a conservational bicategory, if $e\colon A \longrightarrow B$, $e'\colon A' \longrightarrow B'$ are strong liberal, so is $e \times e'\colon A \times A' \longrightarrow B \times B'$.* This follows from the fact that $e \times A'$, $B \times e'$ are pullbacks of e, e' along projections, and $e \times e'$ is isomorphic to their composite (2.12).

2.22. An object A of a bicategory $\mathcal{K}$ is called *groupoidal* when every 2-cell $\sigma\colon u \Rightarrow v\colon X \longrightarrow A$ is invertible. This is the same as saying that (the unique up to isomorphism) $A \longrightarrow 1$ is conservative (provided $\mathcal{K}$ has a terminal object 1). Write $\mathrm{G}\mathcal{K}$ for the full subbicategory of $\mathcal{K}$ consisting of the groupoidal objects.

2.23. If $\mathcal{K}$ satisfies (2.18) then the inclusion of $\mathrm{G}\mathcal{K}$ in $\mathcal{K}$ has a left biadjoint $\pi\colon \mathcal{K} \longrightarrow \mathrm{G}\mathcal{K}$ whose value at A is given by factoring $A \longrightarrow 1$ as a strong liberal $A \longrightarrow \pi A$ followed by a conservative $\pi A \longrightarrow 1$.

2.24. Proposition. *If $\mathcal{K}$ is conservational then the homomorphism $\pi\colon \mathcal{K} \longrightarrow \mathrm{G}\mathcal{K}$ preserves products.*

Proof. By (2.21), $A \times B \longrightarrow \pi A \times \pi B$ is slib; and by (2.4) the composite of the pair $\pi A \times \pi B \longrightarrow 1 \times 1 \xrightarrow{\sim} 1$ is conservative. So $\pi(A \times B) \xrightarrow{\sim} \pi A \times \pi B$. □

2.25. Definition. A bicategory $\mathcal{K}$ is called *faithfully conservational* when it is conservational and every conservative arrow is faithful (2.6). In this case, every groupoidal object A is *discrete* in the sense that each hom category $\mathcal{K}(X, A)$ is equivalent to a discrete category. We write $\mathrm{D}\mathcal{K}$ instead of $\mathrm{G}\mathcal{K}$.

2.26. There is a *slice bicategory* $\mathcal{K}/U$ obtained from any bicategory $\mathcal{K}$ and an object U thereof. The hom category $(\mathcal{K}/U)(u,v)$ is the pseudoequalizer (1.9) of the two functors $u!, v\circ - \colon \mathcal{K}(A,B) \longrightarrow \mathcal{K}(A,U)$ which are constant at u and composition with v, respectively. The arrows $(f,\nu)\colon u \longrightarrow v$ of $\mathcal{K}/U$ are pictured as triangles:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "u"'] & \arrow[d, phantom, "\underset{\nu}{\cong}"] & B \arrow[dl, "v"] \\ & U & \end{tikzcd}$$

The composition of $\mathcal{K}/U$ is obtained by pasting triangles in the obvious way.

2.27. Proposition. *If $\mathcal{K}$ is (faithfully) conservational then so is each slice $\mathcal{K}/U$. The forgetful homomorphism $\mathcal{K}/U \longrightarrow \mathcal{K}$ preserves and reflects pseudopullbacks, conservatives and strong liberals.*

Proof. The pseudopullback of $(f,\nu)\colon u \longrightarrow w$ and $(g,\mu)\colon v \longrightarrow w$ in $\mathcal{K}/U$ is just obtained from the pseudopullback of f and g. A terminal object for $\mathcal{K}/U$ is the identity $1_U\colon U \longrightarrow U$. The cotensor product of the object $u \in \mathcal{K}/U$ with $\mathbf{2}$ is either of the isomorphic arrows $K_u \longrightarrow U$ coming from the invertee K_u of u. So $\mathcal{K}/U$ is finitely complete. An arrow $(f,\nu)\colon u \longrightarrow v$ with f conservative in $\mathcal{K}$ is clearly conservative in $\mathcal{K}/U$. An arrow $(f,\nu)\colon u \longrightarrow v$ with f a coinverter in $\mathcal{K}$ is clearly a coinverter in $\mathcal{K}/U$. This last result implies another similar one for strong liberals by (2.17) and (2.19).

Now given an arrow $(f,\nu)\colon u \longrightarrow v$, factorize f as $f \cong je$ with j conservative and e slib in $\mathcal{K}$, and then we obtain the desired factorization $u \longrightarrow vj \longrightarrow v$ of (f,ν) in $\mathcal{K}/U$. It follows that (f,ν) is slib in $\mathcal{K}/U$ iff f is in $\mathcal{K}$. It then follows that a pseudopullback of a slib in $\mathcal{K}/U$ is slib. Since the arrow $K_u \longrightarrow K_v$ induced on invertees by $e\colon A \longrightarrow B$ can be obtained as the pseudopullback of $e^{\mathbf{2}}$ along $K_v \longrightarrow B^{\mathbf{2}}$, we also have (2.18)(iii) for $\mathcal{K}/U$. □

2.28. Proposition. *Suppose* M *is a doctrine* [22]*(2.10) on the (faithfully) conservational bicategory $\mathcal{K}$, with underlying endo-homomorphism which preserves strong liberal arrows. Then the bicategory $\mathcal{K}^{\mathrm{M}}$* [22]*(2.19) of algebras for* M *is (faithfully) conservational. The forgetful homomorphism $\mathcal{K}^{\mathrm{M}} \longrightarrow \mathcal{K}$ preserves and reflects finite limits and strong liberal arrows.*

Proof. Straightforward. □

§3. Examples and Non-Examples

3.1. Proposition. *An arrow $e\colon A \longrightarrow B$ in Cat is liberal iff each object $b \in B$ is a retract of an object of the form ea for some $a \in A$.*

Proof. Suppose $e\colon A \longrightarrow B$ satisfies the condition. Let $\sigma\colon u \Rightarrow v\colon B \longrightarrow C$ be a natural transformation for which σe is invertible. Then each component σ_{ea} is invertible. By naturality, each σ_b is a retract of a σ_{ea}, and so is invertible itself. Therefore σ is invertible and it follows that e is liberal.

Conversely, suppose $e\colon A \longrightarrow B$ is liberal. Then its coinvertee is invertible. Since the usual inclusion $Cat \longrightarrow Mod$ preserves weighted colimits [3], the coinvertee of e in Mod is invertible. Hence e is "Cauchy dense" in Mod in the sense of [23]. So the canonical function

$$\int^a B(b, ea) \times B(ea, b') \longrightarrow B(b, b')$$

is surjective [23](Proposition 1). So there exists $b \longrightarrow ea$, $ea \longrightarrow b$ with composite the identity of b, as required. $\square$

3.2. Let Cat_{cc} denote the full subbicategory of Cat consisting of those small categories in which idempotents split.

3.3. Remark. *Neither Cat nor Cat_{cc} is conservational.*

Proof. Let A be the free category generated by the graph

$$a \xrightarrow{\theta} b \xleftarrow[\alpha]{} c \xrightarrow{\phi} d.$$

Let X be the category generated by the graph

$$x \xrightarrow{\rho} y \xrightarrow{\sigma} x$$

subject to the relation $\sigma\rho = 1_x$. The functor $f\colon A \longrightarrow X$, given by

$$f\theta = \rho, f\alpha = 1_y, f\phi = \sigma$$

is in Cat_{cc}. The functor $e\colon A \longrightarrow \mathbf{3}$, taking a, b, c, d to $0, 1, 1, 2$ is a coinverter for the invertee of f. If Cat or Cat_{cc} were conservational, we would have, by Theorem 2.19, a factorization $f \cong je$ with $j\colon \mathbf{3} \longrightarrow X$ conservative. But such a j must take $0, 1, 2$ to x, y, x and so invert the non invertible $0 \longrightarrow 2$ in $\mathbf{3}$, contrary to j being conservative. $\square$

3.4. Proposition. *The bicategory Cat_{cc}^{op} is (faithfully) conservational while Cat^{op} fails only to satisfy (2.18)(ii). The strong conservative arrows $j\colon X \longrightarrow Y$ in Cat are the fully faithful functors such that each retract of an object of the form jx with $x \in X$ is isomorphic to an object of the same form. The strong conservative arrows in Cat_{cc} are the fully faithful functors.*

Proof. We begin by showing that each fully faithful $j\colon X \longrightarrow Y$, "closed under retracts" (as in the Proposition), is scon in Cat. Take a square in Cat as in (2.13) with e liberal. In order to define w as in (2.13), take $b \in B$; this is a retract of some ea with $a \in A$, by (3.1). Then vb is a retract of $vea \cong jua$. By the condition on j, we have $vb \cong jwb$ for some $wb \in X$. This choice defines w on objects. The definition of w on arrows is forced since j is fully faithful. This clearly gives a factorization of the square which is unique up to isomorphism. Since *Cat* admits tensoring with $\mathbf{2}$, we have shown (2.13) that each such j is scon.

If idempotents split in X, notice that any fully faithful functor $j\colon X \longrightarrow Y$ is automatically retract closed (since there is an idempotent in X which maps to the idempotent generated by any retract of a jx). So $j\colon X \longrightarrow Y$ in Cat_{cc} is scon if it is fully faithful.

Both *Cat* and Cat_{cc} are finitely cocomplete bicategories. The former follows from the fact that the 2-category *Cat* admits all weighted pseudocolimits [21], and these provide the bicategorical colimits required. The latter uses the idempotent ("Cauchy") completion homomorphism $\mathcal{Q}\colon Cat \longrightarrow Cat_{cc}$ to turn the constructions in *Cat* into those for Cat_{cc}.

Thus we have (2.18)(i) for Cat^{op} and Cat_{cc}^{op}. To prove (2.18)(iv), take any functor $f\colon A \longrightarrow Y$. Let X be the full subcategory of Y consisting of all retracts of objects of the form $fa, a \in A$. Then we get $f = je$ where j is the inclusion of X in Y and e is liberal in *Cat*. From the beginning of this proof we see that j is scon in *Cat* and in Cat_{cc}. So we have (2.18)(iv) as required.

We can now see that scons in *Cat* and Cat_{cc} are of the type claimed in the Proposition. For, take any scon $f\colon A \longrightarrow Y$ and factorize it $f = je$ as above. Since f, j are scon, so is e (by the dual of (2.12)). Since e is also liberal, it is an equivalence. But j is a scon of the claimed type. This "type" clearly includes equivalences and is closed under composition. So $f = je$ is of that type.

Tensoring with $\mathbf{2}$ in *Cat* is simply given by cartesian product $\mathbf{2} \times -$. This is also true in Cat_{cc} (since Cat_{cc} is closed under exponentiation in *Cat* which implies $\mathcal{Q}$ preserves finite products). With our characterization of scons, it is clear that, if $j\colon X \longrightarrow Y$ is scon, so is $\mathbf{2} \times j\colon \mathbf{2} \times X \longrightarrow \mathbf{2} \times Y$. This proves (2.18)(iii).

It remains to prove (2.18)(ii). The "inclusion" $Cat \longrightarrow Mod$ preserves pseudopushouts [3] and takes each $A \longrightarrow \mathcal{Q}A$ to an equivalence. So any pseudopushout

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & Y \arrow[d, "h"] \\ A \arrow[r, "g"'] & P \end{tikzcd}$$

in *Cat* or Cat_{cc} remains so in *Mod*. If j is fully faithful then the adjunction $j \dashv j^*$ has invertible unit $1_X \cong j^*j$. Hence there exists a module $m\colon P \longrightarrow A$ with $1_A \cong mg$ and $mh \cong fj^*$. Using the universal property of the pseudopushout, we obtain a 2-cell $gm \Rightarrow 1_P$ and prove $m \cong g^*$. So $g \dashv g^*$ has invertible unit, in other words g is fully faithful. This completes the proof in the case of Cat_{cc}.

To see that *Cat* does not satisfy (2.18)(ii), we use the construction of pseudopushout given in (1.15). Take Y to be the category generated by the graph $x \longrightarrow y \longrightarrow x'$, and let A be the category generated by the graph $a \longrightarrow b \longrightarrow a$ subject to the relation that the composite is the identity of a. Let X be the full subcategory of Y consisting of x, x'. Let $f\colon X \longrightarrow A$ take $x \longrightarrow x'$ to $a \longrightarrow b$. Then the inclusion $j\colon X \longrightarrow Y$ is scon in *Cat*, while y is a retract of a in P not isomorphic to either object of A. So the inclusion $g\colon A \longrightarrow P$ is not scon. $\square$

3.5. Notice that $\mathcal{K}$ is conservational iff $\mathcal{K}^{co}$ is.

3.6. Proposition. *The class of functors $e\colon A \longrightarrow B$ in Cat satisfying the three conditions:*

(i) *each object of B is isomorphic to one of the form ea for some $a \in A$;*
(ii) *each $\beta \in B(ea, ea')$ has the form $\beta = (e\alpha)(e\sigma)^{-1}$ where α, σ are arrows of A and σ is inverted by e;*
(iii) *if $(e\alpha)(e\sigma)^{-1} = (e\alpha')(e\sigma')^{-1}$ then there exist arrows γ, γ' of A inverted by e such that $\sigma\gamma = \sigma'\gamma'$ and $\alpha\gamma = \alpha'\gamma'$;*

is contained in the class of strong liberal arrows in Cat and is closed under pseudopullback.

Proof. Consider a square in *Cat* as in (2.13) where e is in the class described in the proposition and j is conservative. We need to define a diagonal w for the square. For each $b \in B$, choose $a \in A$ and an isomorphism $b \cong ea$ (which is possible by (i)). Define w on objects by $wb = ua$. Each arrow $\beta\colon b \longrightarrow b'$ in B is isomorphic by the chosen isomorphisms to an arrow $ea \longrightarrow ea'$ which, by (ii), has the form $(e\alpha)(e\sigma)^{-1}$. Notice that $e\sigma$ invertible implies $ve\sigma$, and hence $ju\sigma$ invertible; so $u\sigma$ is invertible (using j conservative). So we can define $w\beta = (u\alpha)(u\sigma)^{-1}$ which is independent of choice of α, σ bi (iii). This gives w and $u \cong we$, $jw \cong v$ as desired in (2.13). So e is slib.

The stability under pseudopullbacks is straightforward, and is a worthwhile exercise for the reader. $\square$

3.7. Let *Lex* (resp. *Rex*) denote the subbicategory of *Cat* consisting of the categories which admit finite limits (resp. finite colimits), the functors which preserve these, and all the natural transformations between such functors.

3.8. Proposition. *The bicategories Lex and Rex are both conservational.*

Proof. Notice that taking duals gives an isomorphism $\mathit{Rex}^{co} \cong \mathit{Lex}$. So it suffices by (3.5) to prove that *Lex* is conservational. Limits in this bicategory are formed as in *Cat* (one reason is that the inclusion $\mathit{Lex} \longrightarrow \mathit{Cat}$ has a left biadjoint [24]). Also, the conservative arrows are precisely the conservative functors.

If $f\colon A \longrightarrow Y$ is an arrow of *Lex* then the arrows of A in the invertee of f form a calculus of right fractions [7]. By looking at the construction of the category of fractions for a calculus of fractions [7], we see that the coinverter of this invertee is a functor e in the class specified in Proposition 3.6, the category X has finite limits, the functor $j\colon X \longrightarrow Y$ satisfying $f = je$ is conservative, and e, j preserve finite limits.

In the diagrams of (2.13) in the case of *Cat*, notice that, since j is conservative, the functor w will preserve whatever limits the categories have and $v(\cong jw)$ preserves. It follows from Proposition 3.6 that $e\colon A \longrightarrow X$ in *Lex* satisfying the three conditions are slib in *Lex* and are closed under pseudopullback. Because of the factorization of the last paragraph, the arrows in the class of Proposition 3.6 which are in *Lex* are precisely the slib arrows in *Lex*. Thus we have proved (2.18)(i)(ii),(iv) for *Lex*.

In fact, any arrow $e\colon A \longrightarrow B$ in *Lex* satisfies condition (iii) of Proposition 3.6 automatically. Just take the pullback τ, τ' of σ, σ', then take the equalizer ρ of $\alpha\sigma$, $\alpha\sigma'$, and finally put $\gamma = \tau\rho$, $\gamma' = \tau'\rho$.

It remains to show that, if $e\colon A \longrightarrow B$ in *Lex* satisfies conditions (3.6)(i) and (ii), then so does $e^{\mathbf{2}}\colon A^{\mathbf{2}} \longrightarrow B^{\mathbf{2}}$. For (i), take an object $\beta\colon b \longrightarrow b'$ in $B^{\mathbf{2}}$. By (i) for e we have $ea \cong b \longrightarrow b' \cong ea'$, which by (ii) for e, is of the form $(e\alpha)(e\sigma)^{-1}$; so β is isomorphic to $e\alpha$ as objects of $B^{\mathbf{2}}$. To prove (ii) for $e^{\mathbf{2}}$, take a commutative square in B of the form

$$\begin{tikzcd} ea \arrow[r, "\beta"] \arrow[d, "e\xi"'] & ex \arrow[d, "e\eta"] \\ ea' \arrow[r, "\beta'"'] & ex'. \end{tikzcd}$$

Apply (ii) for e to obtain $\beta' = (e\alpha')(e\sigma')^{-1}$. Let σ_1, ξ_1 form a pullback for σ', ξ and let α_1, η_1 form a pullback for α', η. Then $e\alpha \cdot e\xi_1 = e\eta.\beta.e\sigma_1$, so there exists a unique β_1 with $e\eta_1 \cdot \beta_1 = e\xi_1$, $e\alpha_1 \cdot \beta_1 = \beta \cdot e\sigma_1$. Apply (ii) for e to obtain $\beta_1 = (e\alpha)(e\sigma)^{-1}$. Apply (iii) for e to obtain ρ, ρ' inverted by e and such that $\sigma\rho = 1 \cdot \rho'$, $\eta_1\alpha\rho = \xi_1\rho'$. Then we have arrows $(\sigma_1\rho', \sigma')\colon \eta_1\alpha\rho \longrightarrow \xi$, $(\alpha_1\alpha\rho, \alpha')\colon \eta_1\alpha\rho \longrightarrow \eta$ in $B^{\mathbf{2}}$, the former inverted by $e^{\mathbf{2}}$, with

$$(\beta, \beta') = e^{\mathbf{2}}(\alpha_1\alpha\rho, \alpha') \cdot [e^{\mathbf{2}}(\sigma_1\rho', \sigma')]^{-1}$$

as desired. $\square$

§4. Discrete Fibrations

We begin by reviewing a few concepts from [22] and [24].

4.1. Consider a span $(p, E, q)\colon B \longrightarrow A$ (sometimes abbreviated to $E\colon B \longrightarrow A$) from B to A in the bicategory $\mathcal{K}$; that is, an ordered pair of arrows in $\mathcal{K}$ displayed diagrammatically as follows:

$$\begin{tikzcd} & E \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A & & B \end{tikzcd}$$

A 2-cell $\chi\colon e' \Rightarrow e\colon K \longrightarrow E$ is said to be *left cartesian* (with respect to our span) when:

(i) $q\chi$ is invertible;

(ii) for each triple (g, ξ, α) consisting of an arrow $g\colon L \longrightarrow K$ and 2-cells $\xi\colon e'' \Rightarrow eg\colon L \longrightarrow E$, $\alpha\colon pe'' \Rightarrow pe'g$ such that $p\xi = p\chi g \cdot \alpha$, there exists a unique 2-cell $\xi'\colon e'' \Rightarrow e'g$ with $\xi = \chi g \cdot \xi'$ and $p\xi' = \alpha$.

Notice that if χ is left cartesian for our span, then so is χg for any arrow $g\colon L \longrightarrow K$.

4.2. Notice that spans in $\mathcal{K}$ and $\mathcal{K}^{\mathrm{co}}$ coincide. By convention, on re-interpreting a span $(p, E, q)\colon B \longrightarrow A$ of $\mathcal{K}$ in $\mathcal{K}^{\mathrm{co}}$, we will always reverse its orientation. In other words, in $\mathcal{K}^{\mathrm{co}}$ our canonical span becomes $(q, E, p)\colon A \longrightarrow B$ and we redraw it as

$$\begin{tikzcd} & E \arrow[dl, "q"'] \arrow[dr, "p"] & \\ B & & A \end{tikzcd}$$

We say that a 2-cell in $\mathcal{K}$ is *right cartesian* with respect to (p, E, q) precisely when it is left cartesian in $\mathcal{K}^{\mathrm{co}}$ with respect to the corresponding span (q, E, p).

Later on we will see that this convention is related to a close link between weak duality in bicategories of functors and strong duality in bicategories of profunctors.

4.3. *An arrow of spans* $(\kappa, f, \nu)\colon (p, E, q) \longrightarrow (p', E', q')$ *from* B *to* A consists of an arrow $f\colon E \longrightarrow E'$ and isomorphisms $\kappa\colon p'f \cong p$, $\nu\colon q \cong q'f$. A *2-cell* $\theta\colon (\kappa, f, \nu) \Rightarrow (\lambda, g, \mu)$ between such arrows is a 2-cell $\theta\colon f \Rightarrow g$ such that $\kappa = \lambda \cdot p'\theta$ and $q'\theta \cdot \nu = \mu$. Let $\mathrm{Spn}(K)(B, A)$ denote the bicategory of spans so obtained. The convention of (4.2) identifies $\mathrm{Spn}((\mathcal{K}^{\mathrm{co}}))(A, B)$ with $\mathrm{Spn}(\mathcal{K})(B, A)$.

4.4. When $\mathcal{K}$ has products, we can identify each span (p, E, q) with an object $(p, q)\colon E \longrightarrow A \times B$ of the slice bicategory $\mathcal{K}/A \times B$ (2.26), thus we may identify $\mathrm{Spn}(\mathcal{K})(B, A)$ with $\mathcal{K}/A \times B$.

4.5. A span $(p, E, q)\colon B \longrightarrow A$ in $\mathcal{K}$ is called a *left fibration* when the following condition holds:

(left path lifting) for all arrows $e\colon K \longrightarrow E$ and 2-cells $\alpha\colon a \Rightarrow pe\colon K \longrightarrow A$, there exists a left cartesian $\chi\colon e * \alpha \Rightarrow e$ for which there is an isomorphism $a \cong p(e * \alpha)$ whose composite with $p\chi$ is α.

Dually $(p, E, q) \in \mathcal{K}$ is a *right fibration* if it is a left fibration in $\mathcal{K}^{\mathrm{co}}$, or more explicitly when it satisfies:

(right path lifting) for all arrows $e\colon K \longrightarrow E$ and 2-cells $\beta\colon qe \Rightarrow b\colon K \longrightarrow B$, there exists a right cartesian $\zeta\colon e \Rightarrow \beta * e$ for which there is an isomorphism $q(\beta * e) \cong b$ whose composite with $q\zeta$ is β.

4.6. A span $(p, E, q)\colon B \longrightarrow A$ is called a *discrete fibration* from B to A if it is both a left and right fibration and discrete (2.25) as an object of the bicategory $\mathrm{Spn}(\mathcal{K})(B, A)$. The discreteness of (p, E, q) in $\mathrm{Spn}(\mathcal{K})(B, A)$ may be more explicitly expressed as follows:

(discreteness) for all $\xi, \eta\colon e \Rightarrow e'\colon K \longrightarrow E$, if $p\xi = p\eta$ with $p\xi$, $q\xi$ invertible then $\xi = \eta$ with ξ invertible.

4.7. Proposition. *A span (p, E, q) is a discrete fibration iff the following three conditions hold:*

(unique left path lifting) for all arrows $e\colon K \longrightarrow E$ and 2-cells $\alpha\colon a \Rightarrow pe$, the category, whose objects are pairs $(\chi\colon e' \Rightarrow e, \nu\colon a \cong pe')$ with $p\chi \cdot \nu = \alpha$ and $q\chi$ invertible, and whose arrows are the obvious ones, is essentially discrete (i.e. an equivalence relation) and nonempty;

(unique right path lifting) for all arrows $e\colon K \longrightarrow E$ and 2-cells $\beta\colon qe \Rightarrow b$, the category, whose objects are pairs $(\zeta\colon e \Rightarrow e', \mu\colon qe' \cong b)$ with $\mu \cdot q\zeta = \beta$ and $p\zeta$ invertible, and whose arrows are the obvious ones, is essentially discrete and nonempty;

(factorization) each 2-cell $\xi\colon e' \Rightarrow e''\colon K \longrightarrow E$ is a composite $\xi = \chi \cdot \zeta$ where $q\chi$ and $p\zeta$ are invertible.

Proof. Straightforward. □

4.8. If (p', E', q') satisfies the discreteness condition of (4.6) then there is at most one 2-cell $\theta\colon (\kappa, f, \nu) \Rightarrow (\lambda, g, \mu)\colon (p, E, q) \longrightarrow (p', E', q')$; and, if there is one, it is invertible. Any arrow of spans between discrete fibrations preserves left and right cartesian 2-cells.

4.9. Write $\mathrm{DFib}(\mathcal{K})(B, A)$ for the category whose objects are the discrete fibrations (p, E, q) from B to A and whose arrows are the isomorphism classes $[\kappa, f, \nu]$ of arrows (κ, f, ν) of spans between such.

4.10. Proposition [20], [22]. *Any comma object of a pair $f\colon A \longrightarrow C$, $g\colon B \longrightarrow C$ of arrows in any bicategory gives a discrete fibration $(d_0, f \downarrow g, d_1)\colon B \longrightarrow A$. In Cat, every discrete fibration arises this way.* □

4.11. Suppose that $\mathcal{K}$ has whatever finite limits are needed. For any span (p, E, q), the arrow $d_1\colon A \downarrow p \longrightarrow E$ has a right adjoint $i\colon E \longrightarrow A \downarrow p$ (or i_p when confusion is likely), with isomorphisms $\varepsilon\colon d_1 i \cong 1_E$, this being the counit, and $\tau\colon d_0 i \cong p$. All this follows from the universal property of the comma object $A \downarrow p$.

The objects E, $A \downarrow p$ have canonical interpretations as objects of $\mathcal{K}/A$, namely $p\colon E \longrightarrow A$ and $d_0\colon A \downarrow p \longrightarrow A$, which we shall assume unless otherwise stated. Then the pair (i, τ) becomes an arrow from E to $A \downarrow p$ in $\mathcal{K}/A$, and again if we talk about the arrow i in $\mathcal{K}/A$ we will assume that it is accompanied by the isomorphism τ.

Notice that the canonical (strict) homomorphism $\mathcal{K}/A \longrightarrow \mathcal{K}$ takes an adjunction $\varepsilon, \eta\colon (f, \mu) \dashv (u, \nu)$ in $\mathcal{K}/A$ to an adjunction $\varepsilon, \eta\colon f \dashv u$ in $\mathcal{K}$. Conversely if we are given arrows (f, μ) and (u, ν) and a 2-cell $\varepsilon\colon (f, \mu)(u, \nu) \Rightarrow 1$ in $\mathcal{K}/A$ then $(f, \mu) \dashv (u, \nu)$ with counit ε iff $f \dashv u$ with counit ε in $\mathcal{K}$.

4.12. Proposition. *A span (p, E, q) is a left fibration iff the arrow $(i, \tau)\colon E \longrightarrow A{\downarrow}p$ has a right adjoint $(r, \kappa)\colon A \downarrow p \longrightarrow E$ in $\mathcal{K}/A$ with counit $\varepsilon\colon ir \Rightarrow 1_{A\downarrow p}$ such that $qd_1\varepsilon$ is invertible.*

Proof.
" $\Leftarrow$" Suppose we have an adjoint $r\colon A \downarrow p \longrightarrow E$ as in Proposition. By applying d_1 to the counit $\varepsilon\colon ir \Rightarrow 1_{A\downarrow p}$ and using $1_E \cong d_1 i$, we obtain a 2-cell $\chi\colon r \Rightarrow d_1$. It can be seen, using the adjunction $i \dashv r$ and the fact that $qd_1\varepsilon$ is invertible, that χ is a left cartesian arrow. Furthermore, the rule for ε as a 2-cell of $\mathcal{K}/A$ ensures that the isomorphism $\kappa\colon d_0 \cong pr$ composed with $p\chi$ is equal to $\lambda\colon d_0 \Rightarrow pd_1$, the canonical 2-cell of the comma object $A \downarrow p$. Now consider any 2-cell $\alpha\colon a \Rightarrow pe\colon K \Rightarrow A$, which, by the comma object property, is isomorphic to $\lambda c\colon d_0 c \Rightarrow pd_1 c$ with $d_1 c \cong e$ for some arrow $c\colon K \longrightarrow A \downarrow p$. But we have already shown that χ is a left path lift of λ, therefore it follows that the composite of $\chi c\colon rc \Rightarrow d_1 c$ and $d_1 c \cong e$ is a left path lift of α.
"$\Rightarrow$" Conversely suppose the left path lifting property holds. We have the canonical 2-cell $\lambda\colon d_0 \Rightarrow pd_1\colon A{\downarrow}p \longrightarrow A$ to which we apply left path lifting to get a left cartesian $\chi\colon r \Rightarrow d_1\colon A \downarrow p \longrightarrow E$ and an isomorphism $\kappa\colon pr \cong d_0$ (which makes r into an arrow from E to $A \downarrow p$ in $\mathcal{K}/A$). The universal property of the comma object $A \downarrow p$ implies that for fixed arrows $e\colon K \longrightarrow E$, $e'\colon K \longrightarrow A \downarrow p$, 2-cells $\gamma\colon ie \Rightarrow e'$ correspond to pairs $\hat{\gamma}_0\colon pe \Rightarrow d_0 e'$, $\hat{\gamma}_1\colon pe \Rightarrow pd_1 e'$ such that $\lambda e' \cdot \gamma_0 = \gamma_1$. These in turn, using the left cartesian-ness of χ and the the isomorphism $\kappa\colon pr \cong d_0$, correspond to 2-cells $\hat{\gamma}\colon e \Rightarrow re'$.

Now define a 2-cell $\varepsilon\colon ir \Rightarrow 1_{A\downarrow p}$ in $\mathcal{K}/A$, using the universal property of $A \downarrow p$, to be the unique ε such that $d_0\varepsilon$, $d_1\varepsilon$ are equal to the composites $d_0 ir \cong pr \cong d_0$ and $d_1 ir \cong r \Rightarrow d_1$ respectively. In particular this definition ensures that $qd_1\varepsilon$ is invertible since χ is left cartesian. It is now straightforward to show that if the 2-cell $\hat{\gamma}\colon e \Rightarrow re'$ corresponds to $\gamma\colon ie \Rightarrow e'$ under the bijection of the last paragraph then it is the unique such 2-cell satisfying $\gamma = \varepsilon \cdot i\hat{\gamma}$. In other words $i \dashv r$ with counit ε in $\mathcal{K}$, and ε is a suitable 2-cell in $\mathcal{K}/A$, therefore, by (4.11), $(i, \tau) \dashv (r, \kappa)$ in $\mathcal{K}/A$ with counit ε such that $qd_1\varepsilon$ is invertible as required. □

4.13. If (p, E, q) is a left fibration then the unit η of the adjunction $i_p \dashv r$ is an isomorphism. We show this by first noting that one of the triangle identities for $i_p \dashv r$ implies that $\chi i_p \cdot \eta$ is an isomorphism. Now $1_p = \lambda i_p = p\chi i_p\colon p \Rightarrow p$ so χi_p is a cartesian lift of 1_p, as is any isomorphic 2-cell with domain 1_E, like for instance $\chi i_p \cdot \eta$. But η is the unique map factoring one of these lifts through the other, and so, by the usual essential uniqueness argument, it is an isomorphism.

It follows, by (2.16), that for any left fibration $(p, E, q)\colon B \longrightarrow A$ the right adjoint $r\colon A \downarrow p \longrightarrow E$ to i_p is a strong liberal.

4.14. Notice that, if we know that (p, E, q) is discrete, as in (4.6), then it satisfies the property given in the last Proposition iff $i \dashv r$ **in** $\mathcal{K}$ (with counit ε such that $qd_1\varepsilon$ is invertible) and there exists **some** isomorphism $pr \cong d_0$.

4.15. Clearly, *if the span* $(p, E, q)\colon B \longrightarrow A$ *is a left fibration then so is the derived span* $(p, E, gq)\colon C \longrightarrow A$ *for all arrows* $g\colon B \longrightarrow C$. However, discreteness does **not** carry over under this process.

4.16. *For all arrows* $b\colon K \longrightarrow B$, *if the span* $(p, E, q)\colon B \longrightarrow A$ *is a left fibration then so is* $(p\,\mathrm{pr}_1, E\,{}_q{\times}_b\, K, \mathrm{pr}_2)$.

Proof. Using the isomorphism $qr \cong qd_1$, obtained from $d_1 i \cong 1_E$ and $qd_1\varepsilon\colon qd_1 ir \Rightarrow qd_1$, we may make the adjunction $(i, \tau) \dashv (r, \kappa)$ of Proposition 4.12 into an adjunction in $\mathrm{Spn}(\mathcal{K})(B, A)$, between spans (p, E, q) and $(d_0, A \downarrow p, qd_1)$. Pseudopullback along $b\colon K \longrightarrow B$ determines a homomorphism of bicategories

$$\mathrm{Spn}(\mathcal{K})(B, A) \xrightarrow{\ b^*\ } \mathrm{Spn}(\mathcal{K})(K, A),$$

and $b^*(p, E, q) \simeq (p\,\mathrm{pr}_1, E\,{}_q{\times}_b\, K, \mathrm{pr}_2)$. All that remains is to note that b^* takes our adjunction in $\mathrm{Spn}(\mathcal{K})(B, A)$ to the one in $\mathrm{Spn}(\mathcal{K})(K, A)$ (and therefore in $\mathcal{K}/A$) which is needed to demonstrate that $(p\,\mathrm{pr}_1, E\,{}_q{\times}_b K, \mathrm{pr}_2)$ is a left fibration (cf. Proposition 4.12). $\square$

4.17. Write $\pi_{A,B}$ for the replacement of the homomorphism π when the bicategory $\mathcal{K}$ is replaced by $\mathrm{Spn}(\mathcal{K})(B, A)$ (see (2.23), (2.25) and (4.4)).

4.18. Proposition. *In a conservational bicategory, if* $(p, E, q)\colon B \longrightarrow A$ *is a left fibration then so is* $\pi_{A,B}(p, E, q)\colon B \longrightarrow A$.

Proof. $\pi_{A,B}(p, E, q) = (u, D, v)$ is obtained from (p, E, q) by factoring the arrow $(p, q)\colon E \longrightarrow A \times B$ up to isomorphism as a slib $e\colon E \longrightarrow D$ composed with a conservative $(u, v)\colon D \longrightarrow A \times B$. The squares in the following diagram are pseudopullbacks.

$$\begin{tikzcd}
A \downarrow p \arrow[r, "e'"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\cong"] & A \downarrow u \arrow[r] \arrow[d, "d_1"] \arrow[dr, phantom, "\cong"] & A^2 \times B \arrow[d, "d_1 \times B"] \\
E \arrow[r, "e"'] & D \arrow[r, "{(u, v)}"'] & A \times B
\end{tikzcd}$$

Since $\mathcal{K}$ is conservational, e' is slib. Replacing the left, right hand downward pointing arrows in the above diagram by r (which exists since (p, E, q) is a left fibration), $d_0 \times B$ respectively, and applying (2.13), we obtain a new vertical middle arrow $s\colon A \downarrow u \longrightarrow D$ making the two new squares commute up to isomorphism. This arrow s is our candidate for the required right adjoint to $i_u\colon D \longrightarrow A \downarrow u$.

To produce a counit $\zeta\colon i_u s \Rightarrow 1_{A\downarrow u}$ we shall use the fact that composing on the right with e' is fully faithful (this is because e' is a coinverter by Theorem 2.19, so the functor $\mathcal{K}(e', -)$ is an inverter, and hence fully faithful (1.10)). So we define ζ by the requirement that $\zeta e'\colon i_u s e' \Rightarrow e'$ is the composite of $i_u s e' \cong i_u e r \cong e' i_p r$ and $e'\varepsilon\colon e' i_p r \Rightarrow e'$. Checking that this is indeed a counit for an adjunction $i_u \dashv s$ is straightforward, then the remainder of the conditions on ζ and s, as required by (4.14), follow easily from those for $i_p \dashv r$. $\square$

4.19. Proposition. *In a conservational bicategory, for all spans* $(p,E,q)\colon B\longrightarrow A$ *and* $(u,F,v)\colon C\longrightarrow B$, *the canonical strong liberal* $F\longrightarrow\pi_{B,C}F$ *induces an equivalence* $\pi_{A,C}(E\times_B F)\xrightarrow{\sim}\pi_{A,C}(E\times_B\pi_{B,C}(F))$.

Proof. The two squares in the following diagram are pseudopullbacks.

$$\begin{tikzcd}
E\underset{B}{\times}F \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & E\underset{B}{\times}\pi_{B,C}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & E\times C \arrow[d, "(p,q)\times 1_C"] \\
A\times F \arrow[r] & A\times\pi_{B,C}(F) \arrow[r] & A\times B\times C
\end{tikzcd}$$

The bottom composite is the factorization of $1_A\times(u,v)$ into a slib and cons. Since $\mathcal{K}$ is conservational, the top left arrow is slib. In order to obtain $\pi_{A,C}(E\times_B\pi_{B,C}(F))$, we factor $E\times_B\pi_{B,C}(F)\longrightarrow E\times C\longrightarrow A\times C$ into a slib and a cons. Composing this slib with the slib $E\times_B F\longrightarrow E\times_B\pi_{B,C}(F)$, we obtain a factorization of $E\times_B F\longrightarrow A\times C$ into a slib and a cons. The result follows. $\square$

4.20. Theorem. *Suppose $\mathcal{K}$ is a faithfully conservational bicategory. There is a bicategory*

$$\mathcal{F}=\mathrm{DFib}(\mathcal{K})$$

whose objects are those of $\mathcal{K}$, and whose homcategories are the categories $\mathcal{F}(B,A)=\mathrm{DFib}(\mathcal{K})(B,A)$ *of (4.9). The identity arrow $A\longrightarrow A$ of this bicategory is the discrete fibration $(d_0,A^{\mathbf{2}},d_1)$. The composition functor*

$$\mathcal{F}(B,A)\times\mathcal{F}(C,B)\longrightarrow\mathcal{F}(C,A)$$

takes discrete fibrations $E\colon B\longrightarrow A$, $F\colon C\longrightarrow B$ to $E\circ F=\pi_{A,C}(E\times_B F)\colon A\longrightarrow C$.

Proof. If $(p,E,q)\colon B\longrightarrow A$, $(u,F,v)\colon C\longrightarrow B$ are discrete fibrations then the span $(p\,\mathrm{pr}_1, E\,{}_q\!\times_u F, v\,\mathrm{pr}_2)$ is both a left and right fibration (4.15), (4.16). So $\pi_{A,C}(E\times_B F)$ is also a left and right fibration (4.18); but it is also discrete (2.25), (2.26). Therefore the proposed composite $E\circ F$ is a discrete fibration. By (4.19), we have canonical equivalences between the *ternary composite* $E\circ F\circ G=\pi_{A,D}(E\times_B F\times_C G)$ and the bracketed iterated binary composites $(E\circ F)\circ G$ and $E\circ(F\circ G)$; these provide the canonical associativity isomorphisms $(E\circ F)\circ G\xrightarrow{\cong}E\circ(F\circ G)$ in $\mathcal{F}(B,A)$. Also, we have the equivalence $A^{\mathbf{2}}\times_A E\xrightarrow{\simeq}A{\downarrow}p$ and the slib $r\colon A{\downarrow}p\longrightarrow E$ (2.16), (4.12); so we have a canonical isomorphism $A^{\mathbf{2}}\circ E\xrightarrow{\sim}E$ (and dually, $E\circ A^{\mathbf{2}}\xrightarrow{\sim}E$) in $\mathcal{F}(B,A)$. The coherence properties of these isomorphisms, as required for a bicategory, are easily verified because of the universal nature of the construction. $\square$

4.21. Remark. Recall the convention introduced in (4.2) concerning spans in $\mathcal{K}$ and $\mathcal{K}^{\mathrm{co}}$. We defined right fibrations in terms of left fibrations using that convention, under which we now see that discrete fibrations in $\mathcal{K}$ correspond to discrete fibrations in $\mathcal{K}^{\mathrm{co}}$, by a bijection which *reverses their orientation.* In fact there is a canonical strict isomorphism of bicategories $\mathrm{DFib}(\mathcal{K})^{\mathrm{op}}\cong\mathrm{DFib}(\mathcal{K}^{\mathrm{co}})$.

4.22. There is an *embedding homomorphism* $(\ )_*\colon \mathcal{K}\longrightarrow\mathcal{F}$ which we shall now describe. It is the identity on objects. The arrow $f\colon B\longrightarrow A$ of $\mathcal{K}$ is taken to the discrete fibration $f_* = (d_0, A\downarrow f, d_1)\colon B\longrightarrow A$. The 2-cell $\sigma\colon f\Rightarrow g\colon B\longrightarrow A$ is taken to the (isomorphism class of the) arrow $A\downarrow\sigma\colon (d_0, A\downarrow f, d_1)\longrightarrow(d_0, A\downarrow g, d_1)$ of spans induced, using the comma object property of $A\downarrow g$, by the 2-cell obtained by pasting σ to the square which exhibits $A\downarrow f$ as a comma object. It is straightforward (compare [22]) to see that we do have a homomorphism of bicategories and that it is locally fully faithful.

4.23. Proposition. *For each arrow $f\colon B\longrightarrow A$ in $\mathcal{K}$ the discrete fibration $f^* = (d_0, f\downarrow A, d_1)\colon A\longrightarrow B$ is right adjoint to $f_*\colon B\longrightarrow A$ in $\mathcal{F} = \mathrm{DFib}(\mathcal{K})$.*

Proof. The "composition" arrow $d_1\colon A^{\mathbf{3}}\longrightarrow A^{\mathbf{2}}$ is slib (2.16), (2.17), and it provides an arrow from (d_0, d_2) to (d_0, d_1) in $\mathcal{K}/A\times A$. Applying pseudopullback along $f\times f\colon B\times B\longrightarrow A\times A$, we obtain a slib $(f\downarrow A)\times_A(A\downarrow f)\longrightarrow f\downarrow f$ in $\mathcal{K}/B\times B$. Hence $f^*\circ f_*$ is just $f\downarrow f$. The unit $n\colon B^{\mathbf{2}}\longrightarrow f\downarrow f$ for the adjunction is the arrow of spans induced by $f\lambda\colon fd_0\Rightarrow fd_1\colon B^{\mathbf{2}}\longrightarrow A$.

The composite $f_*\circ f^*$ in $\mathcal{F}$ can be obtained by factoring the composite of the obvious arrow $(A\downarrow f)\times_B(f\downarrow A)\longrightarrow A^{\mathbf{3}}$ and $d_1\colon A^{\mathbf{3}}\longrightarrow A^{\mathbf{2}}$ into a slib $(A\downarrow f)\times_B(f\downarrow A)\longrightarrow f_*\circ f^*$ and a conservative $m\colon f_*\circ f^*\longrightarrow A^{\mathbf{2}}$; this last arrow m is the counit. One of the adjunction triangle conditions follows from the diagram below, which is commutative up to isomorphism, while the other follows by weak duality in $\mathcal{K}$. □

$$\begin{tikzcd}
(A\downarrow f)\underset{B}{\times}B^{\mathbf{2}} \arrow[r] \arrow[dr] \arrow[dd] & (A\downarrow f)\underset{B}{\times}(f\downarrow A)\underset{A}{\times}(A\downarrow f) \arrow[r] \arrow[d] \arrow[drr] & A^{\mathbf{3}}\underset{A}{\times}(A\downarrow f) \arrow[rr] & & A^{\mathbf{2}}\underset{A}{\times}(A\downarrow f) \arrow[dd] \\
 & (A\downarrow f)\underset{B}{\times}(f\downarrow f) \arrow[d] & & f_*\circ f^*\underset{A}{\times}(A\downarrow f) \arrow[ur] \arrow[dll] & \\
A\downarrow f \arrow[r] \arrow[dr] & f_*\circ f^*\circ f_* \arrow[rrr] & & & A\downarrow f \arrow[dlll] \\
 & A\times B & & &
\end{tikzcd}$$

4.24. Remark. Notice that, by taking mates of the structural 2-cells of the homomorphism $(\ )_*\colon\mathcal{K}\longrightarrow\mathcal{F}$ under the adjunctions $f_*\dashv f^*$, we get a homomorphism $(\ )^*\colon\mathcal{K}^{\mathrm{co}}\longrightarrow\mathcal{F}^{\mathrm{op}}$. In fact, the strict isomorphism $\mathcal{F}^{\mathrm{op}} = \mathrm{DFib}(\mathcal{K})^{\mathrm{op}}\cong\mathrm{DFib}(\mathcal{K}^{\mathrm{co}})$ (cf. Remark 4.21) identifies $(\ )^*$ with $(\ )_*\colon\mathcal{K}^{\mathrm{co}}\longrightarrow\mathrm{DFib}(\mathcal{K}^{\mathrm{co}})$.

4.25. Proposition. *For each arrow $(p, E, q)\colon B\longrightarrow A$ in $\mathcal{F}$, there is a canonical isomorphism $E\cong p_*\circ q^*$.*

Proof. Let $P = (A\downarrow p)\times_E(q\downarrow B)$. Since (p, E, q) is a left fibration, there is a right adjoint $r\colon A\downarrow p\longrightarrow E$ to i_p with invertible unit. Following the argument of (4.16) we consider $i_p\dashv r$ as an adjunction in $\mathrm{Spn}(\mathcal{K})(A, B)$, then apply the homomorphism

$$\mathrm{Spn}(\mathcal{K})(A,B)\xrightarrow{\ -\times_B(d_0, B^{\mathbf{2}}, d_1)\ }\mathrm{Spn}(\mathcal{K})(A,B).$$

Notice that we have canonical equivalences $P \simeq (A \downarrow p)\ {}_{qd_1}\!\times_{d_0} B^{\mathbf{2}}$ and $E\ {}_q\!\times_{d_0} B^{\mathbf{2}}$, which we compose with $r \times_E B^{\mathbf{2}}$ to get an arrow $s\colon P \longrightarrow q \downarrow B$ such that $(pd_0, d_1)s \cong (d_0\mathrm{pr}_1, d_1\mathrm{pr}_2)$.

Now r is a right adjoint with invertible unit, as is $r \times_E B^{\mathbf{2}}$, since homomorphisms preserve such things, and it follows that s is as well, so s is slib by (2.16). We also have the slib $r'\colon q \downarrow B \longrightarrow E$, by (4.13), such that $(p,q)r' \cong (pd_0, d_1)$. Finally we construct $p_* \circ q^*$ by factoring $(d_0\mathrm{pr}_1, d_1\mathrm{pr}_2)\colon P \longrightarrow A \times B$ into a slib followed by a cons, but $(d_0\mathrm{pr}_1, d_1\mathrm{pr}_2) \cong (pd_0, d_1)s \cong (p,q)r's$ where $r's$ is slib (being a composite of such) and (p,q) is cons, therefore $E \cong p_* \circ q^*$ as required. □

4.26. Proposition. *Suppose that $(p, E, q)\colon B \longrightarrow A$ is an arrow in $\mathcal{F}$.*

(a) *For any arrow $f\colon C \longrightarrow A$ in $\mathcal{K}$, one has $f^* \circ E \cong (\mathrm{pr}_1, C\ {}_f\!\times_p E, q\,\mathrm{pr}_2)$.*

(b) *For any left fibration $g\colon A \longrightarrow C$ in $\mathcal{K}$, one has $g_* \circ E \cong \pi_{C,B}(gp, E, q)$.*

Proof.

(a) By (4.15) and (4.16) $(\mathrm{pr}_1, C\,{}_f\!\times_p E, q\,\mathrm{pr}_2)$ satisfies left and right lifting; discreteness is obtained using the discreteness of (p, E, q) and the pseudopullback property of $C\ {}_f\!\times_p E$. So $(\mathrm{pr}_1, C\ {}_f\!\times_p E, q\,\mathrm{pr}_2)$ is a discrete fibration. We also have a factorization of $(f \downarrow A) \times_A E \simeq f \downarrow p \longrightarrow C \times B$ as a composite $f \downarrow p \longrightarrow C\ {}_f\!\times_p E \longrightarrow C \times B$ where the first arrow is a right adjoint for the canonical $C\ {}_f\!\times_p E \longrightarrow f \downarrow p$ with invertible unit. So, by (2.16) the result follows.

(b) Using the left fibration properties of (p, E, q) and g , we obtain a right adjoint r to the canonical $i\colon E \longrightarrow C \downarrow gp$ with invertible unit, and this adjunction naturally sits in $\mathrm{Spn}(\mathcal{K})(C, B)$, cf. (4.16). This gives a factorization of $(C \downarrow g) \times_A E \simeq C \downarrow gp$ into the slib r followed by $(gp, q)\colon E \longrightarrow C \times B$. Hence the result. □

§5. Modulated Bicategories

5.1. In this section we introduce some elementary conditions on $\mathcal{K}$ which imply some desirable properties of the bicategory $\mathcal{F} = \mathrm{DFib}(\mathcal{K})$ which we constructed in the last section. In particular we show that $\mathcal{F}$ has lax limits of 1-cells (these are dual to the *collages* of [23]). As corollaries we will see that $(\ )_*\colon \mathcal{K}\longrightarrow\mathcal{M}$ preserves the terminal object, binary products and some cotensors. This, in turn, is enough to show that $\mathcal{F}$ has all *finite local limits* (cf. [23],[5]).

5.2. A bicategory $\mathcal{K}$ is called *comodulated* when it is faithfully conservational and satisfies two extra axioms:

(v) for all pairs of arrows $f\colon A\longrightarrow C$, $g\colon B\longrightarrow C$, both of the canonical projections $d_0\colon f\downarrow g\longrightarrow A$, $d_1\colon f\downarrow g\longrightarrow B$ are strong liberal;

(vi) for all conservative arrows $(a_1,a_2,a_3)\colon X\longrightarrow A_1\times A_2\times A_3$ with factorizations

$$(a_2,a_3)\cong j_1e_1\colon X\longrightarrow A_2\times A_3, \qquad (a_1,a_3)\cong j_2e_2\colon X\longrightarrow A_1\times A_3,$$
$$(a_1,a_2)\cong j_3e_3\colon X\longrightarrow A_1\times A_2$$

into conservative j_i and strong liberal e_i, if $f\colon K\longrightarrow X$ is such that e_1f, e_2f, e_3f are all strong liberal then f is strong liberal.

A bicategory $\mathcal{K}$ is called *modulated* when $\mathcal{K}^{\mathrm{op}}$ is comodulated.

5.3. Remark. The weak dual $\mathcal{K}^{\mathrm{co}}$ of a (co)modulated bicategory $\mathcal{K}$ is again (co)-modulated.

5.4. We refer to [22] and [21] for some background to the following definitions. Given a small bicategory $\mathcal{B}$ let $\mathrm{Bicat}(\mathcal{B}^{\mathrm{co}},\mathcal{F}^{\mathrm{co}})^{\mathrm{co}}$ denote the usual bicategory of *comorphisms*, *optransformations* and *modifications*. The *lax limit* of a comorphism $T\colon \mathcal{B}\longrightarrow\mathcal{F}$ consists of a 0-cell $\mathrm{llim}(T)\in\mathcal{F}$ and an optransformation $l\colon \Delta C\longrightarrow T$,

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "l_B"'] \arrow[dr, phantom, "\overset{l_\beta}{\Leftarrow}"] & C \arrow[d, "l_{B'}"] \\
TB \arrow[r, "T\beta"'] & TB'
\end{tikzcd}$$

composition with which induces an equivalence of categories

$$\mathcal{F}(X,\mathrm{llim}(T))\simeq \mathrm{Bicat}(\mathcal{B}^{\mathrm{co}},\mathcal{F}^{\mathrm{co}})^{\mathrm{co}}(\Delta X,T)$$

for each 0-cell $X\in\mathcal{F}$. For instance, a comonad in $\mathcal{F}$ is no more nor less than a comorphism $\mathbb{1}\longrightarrow\mathcal{F}$; the lax limit of a comonad (if it exists) is its associated object of Eilenberg-Moore coalgebras.

Notice that [23] and [5] make heavy use of lax colimits, which they call *collages*. In fact we are interested in precisely the same *glueing* constructions, but in the context of $\mathcal{F}$ they present themselves as lax limits. This difference stems from the fact that the bicategories of [23] and [5] correspond to the dual $\mathcal{F}^{\mathrm{coop}}$.

5.5. The principal result of this section will show that $\mathcal{F}$ has lax limits of all 1-cells; that is lax limits of *normal* comorphisms with domain $\mathbf{2}$. An lax cone over a discrete fibration $(p,E,q)\colon B \longrightarrow A$ with vertex C comprises a pair of 1-cells $(u_0,F_0,v_0)\colon C \longrightarrow A$ and $(u_1,F_1,v_1)\colon C \longrightarrow B$ accompanied by a 2-cell $[s,\phi]\colon F_0 \Rightarrow E \circ F_1\colon C \longrightarrow A$. In particular notice that the pair $p_*\colon E \longrightarrow A$, $q_*\colon E \longrightarrow B$ may be made into an lax cone over (p,E,q):

$$
(2) \qquad
\begin{tikzcd}
 & E \arrow[dl, "p_*"'] \arrow[dr, "q_*"] \arrow[d, phantom, "\overset{\gamma}{\Rightarrow}"] & \\
A & & B \arrow[ll, "{(p,E,q)}"]
\end{tikzcd}
$$

Here the 2-cell γ is constructed by composing $p_* \circ \eta_q\colon p_* \Rightarrow p_* \circ q^* \circ q_*$, where η_q is the unit of the adjunction $q_* \dashv q^*$ of Proposition 4.23, with the isomorphism obtained by applying $- \circ q_*$ to the 2-cell $p_* \circ q^* \cong E$ of Proposition 4.25.

5.6. Recall from [23],[5] that we say that $\mathcal{F}$ has *local finite limits* if each homset $\mathcal{F}(C,B)$ has all finite limits and furthermore these are preserved by left and right compositions

$$
\mathcal{F}(C,B) \xrightarrow{\ -\circ F\ } \mathcal{F}(D,B)
$$
$$
\mathcal{F}(C,B) \xrightarrow{\ E\circ -\ } \mathcal{F}(C,A)
$$

for all discrete fibrations $(p,E,q)\colon B \longrightarrow A$ and $(u,F,v)\colon D \longrightarrow C$.

5.7. Remark. As an example of the relationship between local limits and lax limits, which will be useful later on, we consider the following. Let $\mathbb{C}$ be a small category and $\mathbf{2}(\mathbb{C})$ the bicategory with two 0-cells 0 and 1, with $\mathbf{2}(\mathbb{C})(1,0) = \mathbb{C}$, and with no other non-trivial 1- or 2-cells. Now suppose that $\mathcal{F}$ has local limits parameterised by $\mathbb{C}$. Then we may calculate the lax limit of any normal comorphism $T\colon \mathbf{2}(\mathbb{C}) \longrightarrow \mathcal{F}$ as that of the 1-cell $(\varprojlim_{\mathbb{C}} T(-))\colon T(0) \longrightarrow T(1)$ (where one of these lax limits exists iff the other one does).

To establish this result it is enough to provide (natural) equivalences between categories of lax cones over these diagrams. An lax cone γ over T with vertex X consists of a pair of 1-cells $\gamma_0\colon X \longrightarrow T(0)$, $\gamma_1\colon X \longrightarrow T(1)$ and a family of 2-cells $\gamma_c\colon \gamma_0 \Rightarrow T(c) \circ \gamma_1$ for $c \in \mathrm{obj}(\mathbb{C})$. The coherence conditions that these 2-cells must satisfy, with respect to the 2-cellular structure of $\mathbf{2}(\mathbb{C})$, amount to saying that they form a cone over the functor $T(-) \circ \gamma_1\colon \mathbb{C} \longrightarrow \mathcal{F}(X, T(0))$ with vertex γ_0.

The appropriate observation at this stage is that the preservation of local limits of type $\mathbb{C}$ by composition ensures that $\varprojlim_{\mathbb{C}}(T(-) \circ \gamma_1) \cong (\varprojlim_{\mathbb{C}} T(-)) \circ \gamma_1$. So the family of 2-cells associated with the lax cone γ, which we know forms a cone over $T(-) \circ \gamma_1$, factors to yield a unique 2-cell $\bar{\gamma}\colon \gamma_0 \Rightarrow (\varprojlim_{\mathbb{C}} T(-)) \circ \gamma_1$. In other words, we get the 2-cell in a lax cone $(\gamma_0, \gamma_1, \bar{\gamma})$ over the 1-cell $\varprojlim_{\mathbb{C}} T(-)$. This construction is clearly reversible, giving us the required equivalences of lax cone categories. $\square$

5.8. Remark. It is straightforward to show that the hom-categories of $\mathcal{F}$ possess finite limits. First notice that there is a reflection from the slice $\mathcal{K}/A \times B$ into its full sub-bicategory of discrete fibrations from B to A. But slices and reflective full sub-bicategories of the finitely complete bicategory $\mathcal{K}$ are also finitely complete. Now the category $\mathcal{F}(B, A)$ was constructed by factoring out isomorphism classes of 1-cells in this (locally discrete) bicategory of discrete fibrations. All that remains for us is to remark that under this process the finite bicategorical limits, which we know exist, factor to corresponding categorical limits in $\mathcal{F}(B, A)$.

We must work harder to prove that these finite limits in the hom-categories of $\mathcal{F}$ are preserved by left and right composition. To do so we will ultimately use the following result:

5.9. Lemma. *If $\mathcal{K}$ is a faithfully conservational bicategory and the homomorphism $(\ )_*\colon \mathcal{K}\longrightarrow\mathcal{F}$ preserves cotensors with the finite category $\mathbb{C}$, then each hom category $\mathcal{F}(B, A)$ has finite limits parameterised by $\mathbb{C}$ and, for each discrete fibration $F\colon C\longrightarrow B$, these are preserved by $-\circ F\colon \mathcal{F}(B, A)\longrightarrow\mathcal{F}(C, A)$.*

Proof. For any object $A \in \mathcal{K}$ there is a cotensor $A^{\mathbb{C}} \in \mathcal{K}$, which comes equipped with a functor $\pi\colon \mathbb{C}\longrightarrow\mathcal{K}(A^{\mathbb{C}}, A)$ inducing an equivalence

(3) $$\mathcal{K}(B, A^{\mathbb{C}}) \xrightarrow[\hat{\pi}]{\simeq} \mathcal{K}(B, A)^{\mathbb{C}}$$

for each object $B \in \mathcal{K}$ (cf. (1.8)). The assumption that this is preserved by $(\ )_*$ simply means that the composite

(4) $$\mathbb{C} \xrightarrow{\pi} \mathcal{K}(A^{\mathbb{C}}, A) \xrightarrow{(\ )_*} \mathcal{F}(A^{\mathbb{C}}, A)$$

induces an equivalence

(5) $$\mathcal{F}(B, A^{\mathbb{C}}) \xrightarrow[\hat{\pi}']{\simeq} \mathcal{F}(B, A)^{\mathbb{C}}.$$

for each object B.

For any category $\mathbb{B}$ let $\mathbf{\Delta}_{\mathbb{B}}\colon \mathbb{B}\longrightarrow\mathbb{B}^{\mathbb{C}}$ denote the usual functor, which takes an object $b \in \mathbb{B}$ to the functor $\mathbf{\Delta}_b\colon \mathbb{C}\longrightarrow\mathbb{B}$ "constant at b". By definition the category $\mathbb{B}$ admits limits of type $\mathbb{C}$ iff $\mathbf{\Delta}_{\mathbb{B}}$ has a right adjoint. The equivalence (3) ensures that there is an (essentially unique) arrow $\Delta\colon A\longrightarrow A^{\mathbb{C}}$ such that $\hat{\pi}(\Delta) \cong \mathbf{\Delta}_{\mathcal{K}(A,A)}(1_A)$. Notice that, since the map $\hat{\pi}'$ in (5) is induced by the composite in (4), we have $\hat{\pi}'(\Delta_*) \cong \mathbf{\Delta}_{\mathcal{F}(A,A)}(1_A)$ and a diagram:

(6) $$\begin{tikzcd} & \mathcal{F}(B,A) \arrow[dl, "\Delta_* \circ -"'] \arrow[dr, "\mathbf{\Delta}_{\mathcal{F}(B,A)}"] \arrow[d, phantom, "\cong"] & \\ \mathcal{F}(B,A^{\mathbb{C}}) \arrow[rr, "\simeq", "\hat{\pi}'"'] & {} & \mathcal{F}(B,A)^{\mathbb{C}} \end{tikzcd}$$

It follows that $\mathbf{\Delta}_{\mathcal{F}(B,A)}$ has a right adjoint iff the functor $\Delta_* \circ -$ does, but Δ_* has a right adjoint Δ^* in $\mathcal{F}$ (see Proposition (4.23)), which gives rise to an adjunction $\Delta_* \circ - \dashv \Delta^* \circ -\colon \mathcal{F}(B, A^{\mathbb{C}})\longrightarrow\mathcal{F}(B, A)$.

It remains to prove the preservation property mentioned in the lemma. Firstly we may recast this to postulate that the mate [12] of the invertible 2-cell

$$
\begin{tikzcd}
\mathcal{F}(B,A) \arrow[r, "\Delta_{\mathcal{F}(B,A)}"] \arrow[d, "-\circ F"'] \arrow[dr, phantom, "\cong"] & \mathcal{F}(B,A)^{\mathbb{C}} \arrow[d, "(-\circ F)^{\mathbb{C}}"] \\
\mathcal{F}(C,A) \arrow[r, "\Delta_{\mathcal{F}(C,A)}"'] & \mathcal{F}(B,A)^{\mathbb{C}},
\end{tikzcd}
\tag{7}
$$

under the adjunctions $\Delta_{\mathcal{F}(B,A)} \dashv \varprojlim_{\mathbb{C}}$ and $\Delta_{\mathcal{F}(C,A)} \dashv \varprojlim_{\mathbb{C}}$, is an isomorphism as well. But the canonical isomorphisms in the squares

$$
\begin{tikzcd}
\mathcal{F}(B,A) \arrow[r, "\Delta_*\circ -"] \arrow[d, "-\circ F"'] \arrow[dr, phantom, "\cong"] & \mathcal{F}(B,A^{\mathbb{C}}) \arrow[d, "-\circ F"] \\
\mathcal{F}(C,A) \arrow[r, "\Delta_*\circ -"'] & \mathcal{F}(C,A^{\mathbb{C}})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(B,A^{\mathbb{C}}) \arrow[r, "\Delta^*\circ -"] \arrow[d, "-\circ F"'] \arrow[dr, phantom, "\cong"] & \mathcal{F}(B,A) \arrow[d, "-\circ F"] \\
\mathcal{F}(C,A^{\mathbb{C}}) \arrow[r, "\Delta^*\circ -"'] & \mathcal{F}(C,A)
\end{tikzcd}
$$

are mates under the adjunctions $\Delta_*\circ - \dashv \Delta^*\circ -$, and the left hand square may be obtained by pasting copies of the triangle in (6) onto the square in (7). It follows, from the right hand square, that the mate of (7) is also an isomorphism. □

5.10. Remark. To obtain the conclusion of the last lemma for all finite limits, it is enough to check that $(\ )_*\colon \mathcal{K}\longrightarrow\mathcal{F}$ preserves the terminal object, binary products and cotensors with $\mathbb{P}$, where $\mathbb{P}$ is the category consisting of a parallel pair of arrows between two different objects:

$$
\mathbb{P} = \boxed{\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}}
$$

This simply reflects the fact that the finite limits in a hom category $\mathcal{F}(B,A)$ may be constructed using its terminal object, binary products and equalizers, when these exist.

5.11. Remark. Suppose we knew the conclusion of lemma (5.9) held for every bicategory of discrete fibrations $\mathcal{F} = \mathrm{DFib}(\mathcal{K})$ constructed from a comodulated bicategory $\mathcal{K}$. Of course $\mathcal{K}^{\mathrm{co}}$ is comodulated iff $\mathcal{K}$ is (5.3) and $\mathcal{F}^{\mathrm{op}} = \mathrm{DFib}(\mathcal{K}^{\mathrm{co}})$ (4.21), so it follows that both $\mathcal{F}$ and $\mathcal{F}^{\mathrm{op}}$ satisfy the conclusion of the lemma. This would serve to demonstrate that $\mathcal{F}$ has local finite limits, as defined in (5.6), since the preservation property of the lemma, for the dual $\mathcal{F}^{\mathrm{op}}$, is no more than the second one given in (5.6) for $\mathcal{F}$ itself.

5.12. *For the remainder of this section we will assume that we are working within a comodulated bicategory $\mathcal{K}$.*

5.13. Proposition. *If (p, E, q) is a left (resp. right) fibration then p (resp. q) is strong liberal.*

Proof. The arrow $i\colon A \longrightarrow A \downarrow p$ has a right adjoint $r\colon A \downarrow p \longrightarrow A$ in $\mathcal{K}/A$ (4.12) with invertible counit, therefore r is slib, by (4.13) and $pr \cong d_0\colon A \downarrow p \longrightarrow A$. Now, by (5.2)(v), d_0 is strong liberal, therefore it follows from (2.12) that p is slib as well. □

5.14. Remark. In what follows it will sometimes aid intuition and make arguments easier to adopt the terminology and informal notation of "generalised elements". When we write $a \in_X A$, and say a is an *object* of A *defined* at X, we are really referring to a 1-cell $a\colon X \longrightarrow A$. Similarly $(\alpha\colon a \Rightarrow a') \in_X A$, or in words α is a *morphism* or *arrow* of A from a to a' defined at X, denotes a 2-cell $\alpha\colon a \Rightarrow a'\colon X \longrightarrow A$.

Of course the foundation on which such a notation rests is simply the ubiquitous bicategorical Yoneda lemma [22] and so its advantages extend beyond saving space and time. This is especially apparent when working with all kinds of limit construction. For instance it is natural to write the generalised objects of a comma object $f \downarrow g$ as triples $(a \in_X A, b \in_X B, \gamma\colon fa \Rightarrow ga \in_X C)$ (or simply $(a, b, \gamma\colon fa \Rightarrow gb)$), with generalised morphisms from (a, b, γ) to (a', b', γ') as pairs $(\alpha \in_X A, \beta \in_X B)$ with $\gamma' \cdot f\alpha = g\beta \cdot \gamma$.

We do not propose to spend any time developing a formal theory supporting this notation. Rather we encourage the reader to gain an intuition for it by translating some of the proofs given in earlier sections.

5.15. Proposition. If $(p, E, q)\colon B \longrightarrow A$ is a discrete fibration and $g\colon A \longrightarrow C$ any right fibration then one has $g_* \circ E \cong \pi_{C,B}(gp, E, q)$ (compare with 4.26(b)).

Proof. First consider the 1-cell $\bar{g}\colon A \downarrow p \longrightarrow C \downarrow gp$ which is induced, via the universal property of $C \downarrow gp$, by the 2-cell $g\lambda\colon gd_1 \Rightarrow gpd_0\colon A \downarrow p \longrightarrow C$. Here $\lambda\colon d_1 \Rightarrow pd_0$ is the canonical 2-cell associated with the comma object $A \downarrow p$. In other words the 1-cell $\bar{g}$ carries an object $(a, e, \alpha\colon a \Rightarrow pe) \in_X A \downarrow p$ to $(ga, e, g\alpha\colon ga \Rightarrow gpe) \in_X C \downarrow gp$, and a morphism $(\tau, \kappa)\colon (a, e, \alpha) \Rightarrow (a', e', \alpha') \in_X A \downarrow p$ to $(g\tau, \kappa) \in_X C \downarrow gp$.

Notice that a 2-cell $(\rho, \chi)\colon (a, e, \alpha) \Rightarrow (a', e', \alpha') \in_X A \downarrow p$ is right cartesian for $\bar{g}$ if $\rho\colon a \Rightarrow a' \in_X A$ is right cartesian for g, so how would we obtain a cartesian lift of a morphism $(\mu, \chi)\colon \bar{g}(a, e, \alpha) \Rightarrow (c, e', \gamma) \in_X C \downarrow gp$. First lift $\mu\colon ga \Rightarrow c \in_X C$ along g to a right cartesian $\rho\colon a \Rightarrow a' \in_X A$ (accompanied by an isomorphism $\tau\colon ga' \cong c$ such that $\tau \cdot g\rho = \mu$), then notice that the pair (μ, χ) is a morphism of $C \downarrow gp$ so we know that $g(p\chi \cdot \alpha) = \gamma \cdot \mu$. Therefore, since ρ is right cartesian for g, $p\chi \cdot \alpha$ factors as $\alpha' \cdot \rho$ for a unique $\alpha'\colon a' \Rightarrow pe'$ with $g\alpha = \gamma \cdot \tau$. Now we have constructed a object (a', e', α'), a right cartesian $(\rho, \chi)\colon (a, e, \alpha) \Rightarrow (a', e', \alpha')$ for $\bar{g}$ and an isomorphism $(\tau, 1_{e'})\colon \bar{g}(a', e', \alpha') \Rightarrow (c, e', \gamma)$ with $(\tau, 1_{e'}) \cdot \bar{g}(\rho, \chi) = (\mu, \chi)$, or in other words this is the required cartesian lift. It follows that $\bar{g}$ is a right fibration and as such Proposition 5.13 implies that it is a strong liberal.

We construct $g_* \circ E$ by factoring the 1-cell $(d_0, qd_1)\colon C \downarrow gp \longrightarrow C \times B$ into a strong liberal $e\colon C \downarrow gp \longrightarrow g_* \circ E$ followed by a conservative $(u, v)\colon g_* \circ E \longrightarrow C \times B$. The composite $(d_0, qd_1)\bar{g}i_p\colon E \longrightarrow C \times B$ is isomorphic to (gp, q) so we have established the proposition if we can prove that $e\bar{g}i_p$ is a strong liberal.

Consider the counit $\varepsilon\colon i_p r \Rightarrow 1_{A\downarrow p}$ of the adjunction $i_p \dashv r$ associated with the left fibration (p, E, q). We know, from the statement of Proposition 4.12, that $qd_1\varepsilon$ is an isomorphism, and the third paragraph of its proof demonstrates that $d_0\varepsilon$ is isomorphic as well. Now the 1-cell $(gd_0, qd_1)\colon A \downarrow p \longrightarrow C \times B$ may be factored as $e\bar{g}\colon A \downarrow p \longrightarrow g_* \circ E$ followed by $(u, v)\colon g_* \circ E \longrightarrow C \times B$ which is conservative so, since we know that $(gd_0, qd_1)\varepsilon$ is isomorphic, it follows that $e\bar{g}\varepsilon$ provides us with an isomorphism between $e\bar{g}i_p r$ and $e\bar{g}$. Of course we know that r (as a right adjoint with invertible unit), e and $\bar{g}$ are all strong liberals therefore we may use the composition and cancellation rules of 2.12 to show that $e\bar{g}i_p$ is slib as required. □

5.16. Construction. The aim of the next few propositions will be to establish that the lax cone in (2) presents E as the lax limit of the 1-cell (p, E, q) in $\mathcal{F}$. The following construction will be crucial to that end.

Suppose (F_0, F_1, s) is a lax cone over E as above, then the composite $E \circ F_1$ is formed by factoring $E \times_B F_1 \longrightarrow C \times A$ into a strong liberal $l\colon E \times_B F_1 \longrightarrow E \circ F_1$ followed by a conservative $(k, w)\colon E \circ F_1 \longrightarrow C \times A$. Now form the pullback:

(8)
$$\begin{tikzcd} H \arrow[r, "t"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner\quad \overset{\nu}{\cong}"] & E \times_B F_1 \arrow[d, "l"] \\ F_0 \arrow[r, "s"'] & E \circ F_1 \end{tikzcd}$$

It is useful to think of H as the collage of the *gamut* (E, F_0, F_1, s) (cf. [22]), although we will not need to prove that result. All we need is the following lemma:

5.17. Proposition. *The span* $(\mathrm{pr}_E t, H, v_0 m)\colon C \longrightarrow E$ *is a discrete fibration, as is* $(u_0 m, H, \mathrm{pr}_{F_1} t)\colon F_1 \longrightarrow A$.

Proof. First notice that the two results we wish to prove are dual, we obtain the second by reinterpreting the first in the bicategory $\mathcal{K}^{\mathrm{co}}$. This works simply because reversing the 2-cells of $\mathcal{K}$ corresponds to reversing the 1-cells of $\mathcal{F}$ (cf. 4.2 and 4.21).

Discreteness: It is important to notice that we have a chain of isomorphisms $v_0 m \cong wsm \cong wlt \cong v_1 \mathrm{pr}_{F_1} t$ so $(\mathrm{pr}_E t, v_0 m) \cong (\mathrm{pr}_E, v_1 \mathrm{pr}_{F_1})t$, and dually an isomorphism $p\,\mathrm{pr}_E t \cong u_0 m$. Now we know that t is conservative, since it is a pullback of the conservative s, but how about $(\mathrm{pr}_E, v_1 \mathrm{pr}_{F_1})$? Suppose that the morphism $\alpha\colon h \Rightarrow h' \in_X H$ has both $\mathrm{pr}_E \alpha$ and $v_1 \mathrm{pr}_{F_1} \alpha$ invertible then, since $q\,\mathrm{pr}_E \cong u_1 \mathrm{pr}_{F_1}$, we also know that $u_1 \mathrm{pr}_{F_1} \alpha$ is an isomorphism. But (u_1, v_1) is conservative which implies that $\mathrm{pr}_{F_1} \alpha$ is an isomorphism, which in turn allows us to infer that α is invertible from the fact that $(\mathrm{pr}_E, \mathrm{pr}_{F_1})$ is conservative as well. Dually $(u_0 m, \mathrm{pr}_{F_1} t)$ is conservative.

Left Fibration: Firstly it is easily seen that a morphism $\sigma\colon h \Rightarrow h' \in_X H$ is left cartesian for $(\mathrm{pr}_E t, H, v_0 m)$ if both of $m\sigma$ and $\mathrm{pr}_{F_1} t\sigma$ are left cartestians for (u_0, F_0, v_0) and (u_1, F_1, v_1) respectively. So how might we lift a morphism $\alpha\colon e \Rightarrow \mathrm{pr}_E th \in_X E$ to such a left cartesian arrow?

(1) Apply $q\colon E \longrightarrow A$ to α and compose the result with the isomorphism $q\,\mathrm{pr}_E th \cong u_1 \mathrm{pr}_{F_1} th$ and then lift the result to obtain a left cartesian arrow $\chi\colon f_1 \Rightarrow \mathrm{pr}_F th$ for

the span (u_1, F_1, v_1) along with an isomorphism $\tau\colon u_1 f_1 \cong qe$ such that the diagram

$$\begin{tikzcd} u_1 f_1 \arrow[r, Rightarrow, "\sim", "\tau"'] \arrow[d, Rightarrow, "u_1\chi"'] & qe \arrow[d, Rightarrow, "q\alpha"] \\ u_1 \,\mathrm{pr}_F th \arrow[r, Rightarrow, "\sim"] & q\,\mathrm{pr}_E th \end{tikzcd}$$

commutes. This of course means that we have formed an object $(e, f_1, \tau) \in_X E \times_B F_1$ and a morphism $(\alpha, \chi)\colon (e, f_1, \tau) \Rightarrow th$.
(2) Apply $p\colon E \longrightarrow A$ to α and compose with the isomorphism obtained by right application of h to $p\,\mathrm{pr}_E t \cong u_0 m$ (cf. proof of discreteness above), then lift the result to obtain a left cartesian arrow $\rho\colon f_0 \Rightarrow mh$ for the span (u_0, F_0, v_0) along with an isomorphism $\kappa\colon u_0 f_0 \cong pe$.
(3) Now s is a map of discrete fibrations so $s\rho\colon sf_0 \Rightarrow smh$ is left cartesian for $(k, E \circ F_1, w)$. From the proof of 4.16 it is clear that $l(\alpha, \chi)$ is also left cartesian for $(k, E \circ F_1, w)$ since χ is left cartesian for (u_1, F_1, v_1). Composing the isomorphisms $ksf_0 \cong u_0 f_0$, $\kappa\colon u_0 f_0 \cong pe$ and $pe \cong kl(e, f_1, \tau)$ we get an isomorphism κ' which is easily shown to make the square

$$\begin{tikzcd} ksf_0 \arrow[r, Rightarrow, "\sim", "\kappa'"'] \arrow[d, Rightarrow, "ks\rho"'] & kl(e, f_1, \tau) \arrow[d, Rightarrow, "kl(\alpha,\chi)"] \\ ksmh \arrow[r, Rightarrow, "\sim", "k\nu h"'] & klth \end{tikzcd}$$

commute. So by the universal properties of $s\rho$ and $l(\alpha, \chi)$ there exists an unique isomorphism $\lambda\colon sf_0 \cong l(e, f_1, \tau)$ such that the diagram

$$\begin{tikzcd} sf_0 \arrow[r, Rightarrow, "\sim", "\lambda"'] \arrow[d, Rightarrow, "s\rho"'] & l(e, f_1, \tau) \arrow[d, Rightarrow, "l(\alpha,\chi)"] \\ smh \arrow[r, Rightarrow, "\sim", "\nu h"'] & lth \end{tikzcd} \tag{9}$$

commutes and $k\lambda = \kappa'$.

The isomorphism λ completes the definition of an object $((e, f_0, \tau), f_1, \lambda) \in_X H$ and the commutative square (9) ensures that the the pair $((\alpha, \chi), \rho)$ becomes an arrow $((e, f_1, \tau), f_0, \lambda) \Rightarrow h$. As we mentioned at the beginning of this proof, the fact that χ and ρ are left cartesian (for F_0 and F_1 resp.) implies that $((\alpha, \chi), \rho)$ itself is left cartesian for $(\mathrm{pr}_E t, H, v_0 m)$ and it is clearly a lift of α as required. So $(\mathrm{pr}_E t, H, v_0 m)$ is a left fibration, and dually $(u_0 m, H, \mathrm{pr}_{F_1} t)$ is a right fibration.

Right Fibration: We know that $(u_0 m, H, \mathrm{pr}_{F_1} t)$ and (u_1, F_1, v_1) are both right fibrations, therefore the composite $v_1 \,\mathrm{pr}_{F_1} t\colon H \longrightarrow C$ is a right fibration. We also have a sequence of isomorphisms $v_1 \,\mathrm{pr}_{F_1} t \cong wlt \cong wsm \cong v_0 m$ so $v_0 m\colon H \longrightarrow C$ is a right fibration as well. To lift an arrow $\beta\colon v_1 \,\mathrm{pr}_{F_1} th \Rightarrow c \in_X C$ we first lift it to a right cartesian for (u_1, F_1, v_1) and then lift again (as above) to H thereby obtaining an arrow $\sigma\colon h \Rightarrow h'$ with $\mathrm{pr}_{F_1} t\sigma$, $\mathrm{pr}_E t\sigma$ and $m\sigma$ all right cartesian for F_1,E and F_0

respectively. Therefore $p\,\mathrm{pr}_E t\sigma$ and $u_1\,\mathrm{pr}_{F_1} t\sigma$ are both isomorphisms, but $u_1\mathrm{pr}_{F_1} \cong q\,\mathrm{pr}_E$ so $(p,q)\mathrm{pr}_E t\sigma$ is invertible. The pair (p,q) is conservative which implies that $\mathrm{pr}_E t\sigma$ is isomorphic and so σ is right cartesian for the span $(\mathrm{pr}_E t, H, v_0 m)$. This establishes that $(\mathrm{pr}_E t, H, v_0 m)$ is a right fibration and dually $(u_0 m, H, \mathrm{pr}_{F_1} t)$ is a left fibration. □

5.18. Theorem. The lax cone (p_*, q_*, γ) in diagram (2) displays E as the lax limit of the 1-cell $(p, E, q)\colon B \longrightarrow A$ in $\mathcal{F}$.

Proof. For each 0-cell $C \in \mathcal{F}$ composition with the cone (p_*, q_*, γ) gives a functor

$$\begin{array}{ccc} \mathcal{F}(C,E) & \xrightarrow{\quad\Phi_C\quad} & \mathrm{Bicat}(\mathbb{1}, \mathcal{F}^{\mathrm{co}})^{\mathrm{co}}(\Delta C, (p,E,q)) \\ (f,G,g) & \longmapsto & (p_* \circ G, q_* \circ G, \gamma \circ G) \end{array}$$

which we must prove to be an equivalence, but we have

(a) $(u_0, F_0, v_0) \stackrel{\mathrm{def}}{=} p_* \circ (f,G,g) \cong \pi_{A,C}(pf, G, g)$ by Proposition 4.26(b), since p is a left fibration. Of course this means that there is a strong liberal $e_0\colon G \longrightarrow F_0$ such that $(pf, g) \cong (u_0, v_0)e_0$.

(b) $(u_1, F_1, v_1) \stackrel{\mathrm{def}}{=} q_* \circ (f,G,g) \cong \pi_{B,C}(qf, G, g)$ by Proposition 5.15, since q is a right fibration. This means that there is a strong liberal $e_1\colon G \longrightarrow F_1$ along with an isomorphism $\theta\colon (qf, g) \cong (u_1, v_1)e_1$.

Now consider the diagram

(10)
$$\begin{tikzcd} G \arrow[rr, "{(f,e_1,\theta_B)}"] \arrow[d, "e_0"'] \arrow[drr, phantom, "\xi \atop \cong"] & & E \times_B F_1 \arrow[d, "l"] \\ F_0 \arrow[dr, "{(u_0,v_0)}"'] & & E \circ F_1 \arrow[dl, "{(k,w)}"] \\ & A \times C & \end{tikzcd}$$

where the isomorphism $\theta_B\colon qf \cong u_1 e_1$, comprising part of the data for the top arrow in this diagram, is in fact the second component of the isomorphism $\theta\colon (qf,g) \cong (u_1,v_1)e_1$ in (b) above and ξ is constructed in the obvious way from the various isomorphisms $(k,w)l(f,e_1,\theta_B) \cong (p\,\mathrm{pr}_E, v_1\,\mathrm{pr}_{F_1})(f,e_1,\theta_B) \cong (pf, v_1 e_1)$, $(u_0,v_0)e_0 \cong (pf,g)$ and $v_1 e_1 \cong g$. But e_0 is strong liberal and (k,w) is conservative so our diagram factorises uniquely (up to isomorphism) as

(11)
$$\begin{tikzcd} G \arrow[rr, "{(f,e_1,\theta_B)}"] \arrow[d, "e_0"'] \arrow[drr, phantom, "\cong"] & & E \times_B F_1 \arrow[d, "l"] \\ F_0 \arrow[rr, "s"] \arrow[dr, "{(u_0,v_0)}"'] & \arrow[d, phantom, "\phi \atop \cong"] & E \circ F_1 \arrow[dl, "{(k,w)}"] \\ & A \times C & \end{tikzcd}$$

(cf. 2.13). It is straightforward to check that the isomorphism class $[s,\phi]$ is precisely the composite $\gamma \circ H$. We leave the action of Φ_C on 2-cells $\alpha\colon H \Rightarrow H'\colon C \longrightarrow E$ up to the reader to determine.

Our description of Φ_C, and Proposition 5.17, clearly indicate that Construction 5.16 may be used to provide a potential equivalence inverse:

$$\begin{array}{ccc} \mathrm{Bicat}(\mathbb{1},\mathcal{F}^{\mathrm{co}})^{\mathrm{co}}(\Delta C,(p,E,q)) & \xrightarrow{\quad\Phi'_C\quad} & \mathcal{F}(C,E) \\ (F_0,F_1,s) & \longmapsto & (\mathrm{pr}_E t, H, v_0 m) \end{array}$$

Examining the definition of $(\mathrm{pr}_E t, H, v_0 m)$ reveals that there is a natural way to define the action of Φ'_C on modifications of lax cones, but again we leave the details up to the reader. It remains to provide natural isomorphisms $\Phi_C\Phi'_C \cong 1$ and $\Phi'_C\Phi_C \cong 1$.

$\Phi_C\Phi'_C \cong 1$: Recall the isomorphisms $u_0 m \cong p\,\mathrm{pr}_E t$, $v_0 m \cong v_1 \mathrm{pr}_{F_1} t$, from the "discreteness" portion of the proof of Proposition 5.17, and $q\,\mathrm{pr}_E \cong u_1 \mathrm{pr}_{F_1}$ which give us $(p\,\mathrm{pr}_E t, v_0 m) \cong (u_0,v_0)m$ and $(q\,\mathrm{pr}_E t, v_0 m) \cong (u_1,v_1)\mathrm{pr}_{F_1} t$. But $m\colon H \longrightarrow F_0$, as a pullback of the strong liberal $l\colon E \times_B F_1 \longrightarrow E \circ F_1$, is strong liberal, so $p_* \circ H \cong \pi_{A,C}(p\,\mathrm{pr}_E t, H, v_0 m) \cong (u_0, F_0, v_1)$. Similarly Propositions 5.17 and 5.13 imply that $\mathrm{pr}_{F_1} t$ is a right fibration and thus a strong liberal, therefore $q_* \circ H \cong \pi_{B,C}(q\,\mathrm{pr}_E t, H, v_0 m) \cong (u_1, F_1, v_1)$.

It remains to show that composing these isomorphisms with $\gamma \circ H\colon p_* \circ H \Rightarrow E \circ (q_* \circ H)$ gives us the 2-cell $[s,\phi]$ that we started with. This though is quite straightforward, because in this case diagram (10) reduces to one of the form

$$\begin{tikzcd} H \arrow[rr, "t"] \arrow[d, "m"'] \arrow[drr, phantom, "\cong"] & & E \underset{B}{\times} F_1 \arrow[d, "l"] \\ F_0 \arrow[dr, "{(u_0,v_0)}"'] & & E \circ F_1 \arrow[dl, "{(k,w)}"] \\ & A \times C & \end{tikzcd}$$

which can be shown to factor, as in (11), into a composite of the pullback square in (8) and the triangle associated with the original map of discrete fibrations $[s,\phi]$.

In summary, for each lax cone (F_0,F_1,s) over E we have constructed the required isomorphism $\Phi_C\Phi'_C(F_0,F_1,s) \cong (F_0,F_1,s)$. We leave the naturality of these up to the reader to check.

$\Phi'_C\Phi_C \cong 1$: Starting with a discrete fibration $(f,G,g)\colon C \longrightarrow E$, the construction of $(F_0,F_1,s) \overset{\mathrm{def}}{=} \Phi_C(f,G,g)$ adumbrated above produces, as a byproduct, the factored diagram (11). But H is formed by pulling l back along s, as in (8), so its universal property ensures that we get an (essentially) unique 1-cell $d\colon G \longrightarrow H$ and a factorisation of the square in (11) viz:

(12)
$$\begin{tikzcd}
G \arrow[dr, "d"] \arrow[drr, "{(f,e_1,\theta_B)}"] \arrow[ddr, "e_0"'] & & \\
& H \arrow[r, "t"] \arrow[d, "m"'] \arrow[dr, phantom, "\overset{\nu}{\cong}"] & E \times_B F_1 \arrow[d, "l"] \\
& F_0 \arrow[r, "s"'] & E \circ F_1
\end{tikzcd}$$

Of course we have $v_0e_0 \cong g$ (by definition of e_0, see (a) above), and $\mathrm{pr}_E(f, e, \theta_B) \cong f$ which we compose with the isomorphisms in the triangles of our diagram, then combine to get $\psi\colon (f, g) \cong (\mathrm{pr}_E t, v_0 m)d$. In this way we have constructed a 2-cell $[d, \psi]\colon (f, G, g) \Rightarrow (\mathrm{pr}_E t, H, v_0 m)$ in $\mathcal{F}$, what's more the collection of these form a natural transformation $1 \Rightarrow \Phi'_C \Phi_C$. Now d, as a map of discrete fibrations, is conservative; so in order to prove that $[d, \psi]$ is an isomorphism in $\mathcal{F}$ all that remains is to show that it is strong liberal as well.

It is here that we finally get to apply axiom 5.2(vi) from the definition of a comodulated bicategory. The maps $(p, q)\colon E \longrightarrow A \times B$ and $(\mathrm{pr}_E t, v_0 m)\colon H \longrightarrow E \times C$ are conservative so it follows that $(p\,\mathrm{pr}_E t, q\,\mathrm{pr}_E t, v_0 m)\colon H \longrightarrow A \times B \times C$ is conservative as well. We have already come across the first two of the following factorisations into conservatives following strong liberals (cf. proof of $\Phi_C \Phi'_C \cong 1$):

$$(q\,\mathrm{pr}_E t, v_0 m) \cong (u_0, v_0)m\colon H \to B \times C \qquad (p\,\mathrm{pr}_E t, v_0 m) \cong (u_1, v_1)\mathrm{pr}_{F_1} t\colon H \to A \times C$$
$$(p\,\mathrm{pr}_E t, q\,\mathrm{pr}_E t) \cong (p, q)\mathrm{pr}_E t\colon X \to A \times B$$

For the last one, (p, q) is conservative (by the discreteness of (p, E, q)) and Propositions 5.17 and 5.13 demonstrate that $\mathrm{pr}_E t$ is a left fibration and thus a strong liberal. Composing d with each of these strong liberals in turn, and using the isomorphisms in (12), we get $md \cong e_0$, $\mathrm{pr}_{F_1} td \cong \mathrm{pr}_{F_1}(f, e_1, \theta_B) \cong e_1$ and $\mathrm{pr}_E td \cong \mathrm{pr}_E(f, e_1, \theta_B) \cong f$. Notice though that e_0 and e_1 are strong liberals (by definition) as is f (since it is a left fibration), so we may apply axiom 5.2(vi) and infer that d is a strong liberal as well. $\square$

5.19. Proposition. *The inclusion $()_*\colon \mathcal{K} \longrightarrow \mathcal{F}$ preserves the terminal object of $\mathcal{K}$.*

Proof. The category $\mathcal{F}(A, 1)$ is non-empty, since it contains the discrete fibration $(\square, A, \mathrm{id}_A)$, where $\square\colon A \longrightarrow 1$ is the (essentially) unique 1-cell required by the terminality of 1. Any other discrete fibration $(p, E, q)\colon A \longrightarrow 1$ has $(p, q)\colon E \longrightarrow 1 \times A$ conservative and equivalent to $q\colon E \longrightarrow A$, which is slib by (5.13). So q is an equivalence. It follows that $\mathcal{F}(A, 1)$ is discrete, and so equivalent to $\mathbf{1}$. $\square$

5.20. Proposition. *The inclusion $()_*\colon \mathcal{K} \longrightarrow \mathcal{F}$ preserves the binary products of $\mathcal{K}$.*

Proof. By applying Proposition 5.9 to the result of Proposition 5.19 we establish that $\mathcal{F}$ possesses local terminal objects. Remark 5.8 demonstrates that the terminal object in $\mathcal{F}(B, A)$ is the discrete fibration $(\mathrm{pr}_A, A \times B, \mathrm{pr}_B)$.

Clearly the lax limit of the normal comorphism $\mathbb{1} \coprod \mathbb{1} \longrightarrow \mathcal{F}$, which picks out a pair objects $A, B \in \mathcal{F}$, is no more or less than their product in $\mathcal{F}$. Remark 5.7 implies that we can express this lax limit as that of the terminal 1-cell $(\mathrm{pr}_A, A \times B, \mathrm{pr}_B) \in \mathcal{F}(B, A)$, so by Theorem 5.18 the diagram

$$\begin{tikzcd} & A \times B \arrow[dl, "(\mathrm{pr}_A)_*"'] \arrow[dr, "(\mathrm{pr}_B)_*"] & \\ A & & B \end{tikzcd}$$

displays a product in $\mathcal{F}$ as required. □

5.21. Proposition. *The inclusion* $(\ )_* \colon \mathcal{K} \longrightarrow \mathcal{F}$ *preserve cotensors with* $\mathbb{P}$.

Proof. By applying Proposition 5.9 to the result of Proposition 5.20 we establish that $\mathcal{F}$ possesses all local binary (and therefore finite) products. Remark 5.8 demonstrates that the product of discrete fibrations (p, E, q) and (p', E', q') in $\mathcal{F}(B, A)$ is simply the discrete fibration $(p\, \mathrm{pr}_E, E \times_{A \times B} E', q\, \mathrm{pr}_E)$.

The lax limit of the comorphism $T \colon \mathbb{P} \longrightarrow \mathcal{F}$, which sends both non-identity arrows of $\mathbb{P}$ to the identity 1-cell $(d_0, A^{\mathbf{2}}, d_1) \colon A \longrightarrow A$, is precisely the cotensor of A with $\mathbb{P}$ in $\mathcal{F}$. We have seen that $\mathcal{F}$ has local finite products, so Remark 5.7 can be applied to show that the lax limit of T may be obtained as that of the 1-cell obtained by taking the product of two copies of $(d_0, A^{\mathbf{2}}, d_1)$ in $\mathcal{F}(A, A)$. But the pullback $A^{\mathbf{2}} \times_{A \times A} A^{\mathbf{2}}$, which occurs in the construction of this product of 1-cells, is equivalent to $A^{\mathbb{P}}$, so the lax limit of T is also equivalent to that of the discrete fibration $(d_0, A^{\mathbb{P}}, d_1)$. Theorem 5.18 demonstrates that the lax limit of this final 1-cell (and so of T) in $\mathcal{F}$ is $A^{\mathbb{P}}$ as required. □

5.22. Theorem. $\mathcal{F}$ *has all local finite limits.*

Proof. Finally, applying Proposition 5.9 to the result of Proposition 5.21 we establish that $\mathcal{F}$ also has local equalisers. This completes the list of local limits required for the construction of all finite local limits (cf. Remark 5.10). □

§6. Constructing Modulated Bicategories

6.1. In this section we will examine the principal construction giving rise to modulated bicategories. We start with a bicategory $\mathcal{M}$ (which in practice will be some bicategory of enriched or internal profunctors) possessing:

(i) finite bicategorical coproducts;
(ii) the Kleisli object for each monad;
(iii) local finite colimits (which we should recall are, by definition, preserved by composition on both sides).

We call such a bicategory $\mathcal{M}$ a *finitary cosmos* [23].

An arrow f in $\mathcal{M}$ is called a *map* when it has a right adjoint f^*. We will adopt the convention of using $\eta_f\colon 1 \Rightarrow f^*f$ and $\varepsilon_f\colon ff^* \Rightarrow 1$ to denote the unit and counit of such a map. Let $\mathcal{M}^*$ denote the (locally full) sub-bicategory of $\mathcal{M}$ with the same 0-cells and only those 1-cells which are maps.

6.2. For a monad m on an object A in a bicategory $\mathcal{M}$, we always denote the unit and multiplication by $\eta\colon 1_A \Rightarrow m$ and $\mu\colon mm \Rightarrow m$. An *$m$-algebra into X* is a pair (a, α) where $a\colon A \longrightarrow X$ is a 1-cell and $\alpha\colon am \Rightarrow a$ is a 2-cell satisfying $\alpha \cdot a\eta = 1_a$ *and* $\alpha \cdot a\mu = \alpha \cdot \alpha m$. The category $\mathcal{M}(A,X)^{\mathcal{M}(m,X)}$ of m-algebras into X is the Eilenberg-Moore category for the monad $\mathcal{M}(m,X)$ on the category $\mathcal{M}(A,X)$. An m-algebra (a,α) into K presents that 0-cell as the *Kleisli object* of m if composition with (a,α) provides us with an equivalence

$$\mathcal{M}(K,X) \xrightarrow{\ \sim\ } \mathcal{M}(A,X)^{\mathcal{M}(m,X)}$$

for each 0-cell X. In fact a monad m is precisely a morphism of bicategories $\mathbb{1} \longrightarrow \mathcal{M}$, m-algebras into X are no more than lax cones under that diagram (with vertex X) and the Kleisli object of m (if it exists) is its lax (bi)colimit.

6.3. Recall that an arrow in a category is called an *extremal epic* when it is epic, and any monic into its target, through which it factors, is invertible.

A map $e\colon A \longrightarrow B$ in $\mathcal{M}$ is called *Cauchy dense* when its counit $\varepsilon_e\colon ee^* \Rightarrow 1_B$ is an extremal epic in the category $\mathcal{M}(B,B)$. It was essentially proved in Proposition 1 of [23] that the following conditions are equivalent:

(a) $e\colon A \longrightarrow B$ is Cauchy dense;
(b) the diagram

$$\begin{tikzcd} ee^*ee^* \arrow[r, shift left, "ee^*\varepsilon_e"] \arrow[r, shift right, "\varepsilon_e ee^*"'] & ee^* \arrow[r, "\varepsilon_e"] & 1_B \end{tikzcd}$$

is a coequaliser in $\mathcal{M}(B,B)$;
(c) for all 0-cells X, the functor $\mathcal{M}(e,X)\colon \mathcal{M}(B,X) \longrightarrow \mathcal{M}(A,X)$ is conservative, in other words e is liberal as a 1-cell of $\mathcal{M}$ (cf 2.1, 2.2);
(d) the 0-cell B, 1-cell e and 2-cell $\varepsilon_e e$ provide a Kleisli construction for the monad on A generated by $e \dashv e^*$;
(e) the 0-cell B, 1-cell e^* and 2-cell $e^*\varepsilon_e$ provide an Eilenberg-Moore construction for the monad on A generated by $e \dashv e^*$;
(f) e^* is conservative as a 1-cell of $\mathcal{M}$.

Proof. The equivalence of (b),(c) and (d) was given in Proposition 1 of [23]. The equivalence of (b), (e) and (f) follows from the same proposition, but this time interpreted in the dual bicategory $\mathcal{M}^{\mathrm{op}}$. That (b) implies (a) is trivial, since regular epics are extremal. It remains to see that (a) implies (b).

Since composition preserves local epics, each $m\varepsilon_e\colon mee^* \Rightarrow m$ is epic; so $\mathcal{M}(e,X)$ is faithful for all X when ε_e is epic. Take the coequaliser $\gamma\colon ee^* \Rightarrow n$ of $ee^*\varepsilon_e$ and $\varepsilon_e e^*$ in $\mathcal{M}(B,B)$. Then there exists $\mu\colon n \Rightarrow 1_B$ with $\varepsilon_e = \mu\gamma$. Since $\mathcal{M}(e,B)$ takes the diagram of (b) to a split coequaliser and preserves the coequaliser γ, we have that μe is invertible. But $\mathcal{M}(e,B)$ is faithful, so μ is monic. Finally if ε_e is extremal epic, it factors through the monic μ, which is therefore invertible, proving that (a) implies (b). □

6.4. A map $f\colon A \longrightarrow B$ is called *faithful* when its unit $\eta_f\colon 1_A \Rightarrow f^*f$ is a monic in $\mathcal{M}(A,A)$. Call f *fully faithful* when η_f is invertible.

It was shown in Proposition 1 of [23] that each map $f\colon A \longrightarrow B$ factors up to isomorphism as je where j is a fully faithful map and e is Cauchy dense. The map e is obtained by taking the Kleisli construction of the monad associated with the adjunction $f \dashv f^*$.

6.5. For each arrow $m\colon A \longrightarrow B$, there is a universal diagram

$$\begin{tikzcd}
A \arrow[rr, "m"] \arrow[dr, "\partial_1"'] & \arrow[d, phantom, "\overset{\mu}{\Leftarrow}"] & B \arrow[dl, "\partial_0"] \\
& \langle m, A\rangle &
\end{tikzcd}$$

in $\mathcal{M}$; that is, the cocomma object $\langle m, A\rangle$ of $m, 1_A$ exists. The cospan $(\partial_1, \langle m, A\rangle, \partial_0)$ from A to B is called the *cone on m.* To construct $\langle m, B\rangle$ one takes the Kleisli construction for the monad $\left(\begin{smallmatrix} 1_A & 0 \\ m & 1_B \end{smallmatrix}\right)$ on the coproduct $A \oplus B$ (cf. [29] for details of this matrix notation). It follows that ∂_1 and ∂_0 are fully faithful maps such that $(\partial_1, \partial_0)\colon A \oplus B \longrightarrow \langle m, A\rangle$ is Cauchy dense, the 2-cell $m \Rightarrow \partial_1^*\partial_0$ induced by μ is invertible, and $\partial_0^*\partial_1$ is initial in $\mathcal{M}(B,A)$ (cf. [23] Proposition 1). When $m = 1_A$, note that $\langle m, A\rangle$ is the *tensor product* $\mathbf{2} * A$ of the category $\mathbf{2}$ with the 0-cell $A \in \mathcal{M}$.

More generally, Proposition 1 of [23] gives the construction for collages of morphisms of bicategories from a finite bicategory into $\mathcal{M}$. It was noted that the coprojections into a collage are maps, and a 1-cell out of a collage is a map precisely when its composite with all coprojections is a map. It was also noted, in Proposition 3 of the same paper, that, if $\mathcal{M}$ is a finitary cosmos (cf. 6.1 above) then so is $\mathcal{M}^{\mathrm{op}}$. This follows from the fact, established there, that each collage in $\mathcal{M}$ provides a collage for the corresponding morphism into $\mathcal{M}^{\mathrm{op}}$, by taking right adjoints of the coprojections and forming a lax cocone in $\mathcal{M}^{\mathrm{op}}$. In particular, coproducts are also products and Kleisli objects are also Eilenberg-Moore objects. For this reason we may use the "direct sum" notation $A \oplus B$, for coproducts in $\mathcal{M}$, and matrices to describe 1-cells between such coproducts.

6.6. Let m be a monad on A and (a, α) be an m-algebra presenting K as a Kleisli object for m. If (f, ψ) is an m-algebra into X with f a map, then there is an essentially unique map $h\colon K \longrightarrow X$ with an isomorphism of m-algebras $\tau\colon h(a, \alpha) \cong (f, \psi)$. So the transpose $m \Rightarrow f^*f$ of ψ under the adjunction $f \dashv f^*$ is invertible iff the transpose of $h\alpha$ under the adjunction $ha \dashv a^*h^*$ is an isomorphism. This latter transpose is given by the calculation:

$$
\begin{array}{cl}
ham \xrightarrow{\quad h\alpha \quad} ha & \\
\hline
am \xrightarrow{\ \alpha\ } a \xrightarrow{\ \eta_h a\ } h^*ha & h \dashv h^* \\
\hline
m \xrightarrow{\ \hat{\alpha}\ } a^*a \xrightarrow{\ a^*\eta_h a\ } a^*h^*ha & a \dashv a^*
\end{array}
$$

Since (a, α) presents the Kleisli object of m, we know that $\hat{\alpha}$ is an isomorphism; furthermore a is Cauchy dense, so 6.3 implies that a is conservative and a^* is liberal in $\mathcal{M}$. It follows that the bottom line of our calculation is an isomorphism iff the unit $\eta_h\colon 1 \Rightarrow h^*h$ is invertible. In other words the map h induced by (f, ψ) is fully faithful iff the transpose $\hat{\psi}\colon m \Rightarrow f^*f$ is an isomorphism.

6.7. As we observed (in 6.2) monads in $\mathcal{M}$ are precisely morphisms $\mathbb{1} \longrightarrow \mathcal{M}$, we need to make explicit the notion of optransformation between such morphisms. Let (m, μ_m, η_m) and (n, μ_n, η_n) be monads on A and B respectively. A *monad morphism* $(u, \phi)\colon m \Rightarrow n$ consists of a 1-cell $u\colon A \longrightarrow B$ and a 2-cell $\phi\colon um \Rightarrow nu$ which is compatible with units and multiplications, in the sense that the diagrams

$$
\begin{tikzcd}
& u \arrow[dl, "u\eta_m"'] \arrow[dr, "\eta_n u"] & \\
um \arrow[rr, "\phi"'] & & nu
\end{tikzcd}
\qquad
\begin{tikzcd}
umm \arrow[r, "\phi m"] \arrow[d, "u\mu_m"'] & num \arrow[r, "n\phi"] & nnu \arrow[d, "\mu_n u"] \\
um \arrow[rr, "\phi"'] & & nu
\end{tikzcd}
$$

commute. We will often consider morphisms between monads on the same 0-cell A with $u = 1_A$; in that case we drop explicit mention of 1_A, so long as no confusion could arise by doing so. We say that a monad morphism (u, ϕ) is *strong* if ϕ is an isomorphism. Monad morphisms generalise algebras, for instance an m-algebra into X is precisely a monad morphism from m to the identity monad on X.

Let (a, α) present K as the Kleisli object of m, and (b, β) present L as the Kleisli object of n. Using the compatibility conditions on ϕ, it is easily checked that the pair $(bu, \beta u \cdot b\phi)$ is an m-algebra into L; so there is an essentially unique 1-cell $\overline{(u, \phi)}\colon K \longrightarrow L$ with $\overline{(u, \phi)}(a, \alpha) \cong (bu, \beta u \cdot b\phi)$, and this is a map if u is.

If u is a map then the transpose of $\beta u \cdot b\phi\colon bum \Rightarrow bu$, under the adjunction $bu \dashv u^*b^*$, can be obtained via the following calculation:

$$
\begin{array}{cl}
bum \xrightarrow{\quad b\phi \quad} bnu \xrightarrow{\ \beta u\ } bu & \\
\hline
um \xrightarrow{\quad \phi \quad} nu \xrightarrow{\ \hat{\beta} u\ } b^*bu & b \dashv b^* \\
\hline
m \xrightarrow{\ \eta_u m\ } u^*um \xrightarrow{\ u^*\phi\ } u^*nu \xrightarrow{\ u^*\hat{\beta} u\ } u^*b^*bu & u \dashv u^*
\end{array}
$$

Now (b, β) displays a Kleisli object so $\hat{\beta}$ is an isomorphism; therefore when u is fully faithful and (u, ϕ) is strong, the bottom line of this calculation is an isomorphism. But this is precisely the case considered in 6.6, and it follows that, under these conditions, (u, ϕ) is itself fully faithful.

As an example consider the tensors $\mathbf{2} * A$ in 6.5 above, which were constructed by taking the Kleisli construction for a monad $\left(\begin{smallmatrix} 1_A & 0 \\ 1_A & 1_A \end{smallmatrix}\right)$ on $A \oplus A$. For any 1-cell $j\colon A \longrightarrow B$, the canonical 1-cell $\mathbf{2} * j\colon \mathbf{2} * A \longrightarrow \mathbf{2} * B$ that it induces may be described in terms of a monad morphism. This consists of a 1-cell $\left(\begin{smallmatrix} j & 0 \\ 0 & j \end{smallmatrix}\right)\colon A \oplus A \longrightarrow B \oplus B$ and the isomorphism:

$$\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}\begin{pmatrix} 1_A & 0 \\ 1_A & 1_A \end{pmatrix} \cong \begin{pmatrix} j & 0 \\ j & j \end{pmatrix} \cong \begin{pmatrix} 1_B & 0 \\ 1_B & 1_B \end{pmatrix}\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$$

Now suppose that j is a fully faithful map, then so is $\left(\begin{smallmatrix} j & 0 \\ 0 & j \end{smallmatrix}\right)$ (since its right adjoint is $\left(\begin{smallmatrix} j^* & 0 \\ 0 & j^* \end{smallmatrix}\right)$ with unit $\left(\begin{smallmatrix} \eta_j & 0 \\ 0 & \eta_j \end{smallmatrix}\right)$), so the conditions of the last paragraph apply and therefore $\mathbf{2} * j$ is fully faithful.

6.8. Proposition. *In the bicategory $\mathcal{M}^*$ of maps in a finitary cosmos $\mathcal{M}$, the pseudopushout*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{\phi}{\cong}"] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

exists if f is fully faithful and is preserved by the inclusion $\mathcal{M}^* \longrightarrow \mathcal{M}$ moreover, k is fully faithful.

Proof. First form $B \oplus C$ and consider the matrix

$$m = \begin{pmatrix} 1_B + fg^*gf^* & fg^* \\ gf^* & 1_C \end{pmatrix}$$

representing an endo 1-cell on $B \oplus C$. We define a candidate identity 2-cell, for a monad with underlying 1-cell m, as a matrix

$$\eta = \begin{pmatrix} c_0 & !_{fg^*} \\ !_{gf^*} & 1_{1_C} \end{pmatrix}\colon \begin{pmatrix} 1_B & 0 \\ 0 & 1_C \end{pmatrix} \Longrightarrow \begin{pmatrix} 1_B + fg^*gf^* & fg^* \\ gf^* & 1_C \end{pmatrix}$$

where $c_0\colon 1_B \Rightarrow 1_B + fg^*gf^*$ is the canonical coprojection, and $!_h\colon 0 \Rightarrow h\colon X \longrightarrow Y$ denotes the unique 2-cell into h from the initial object of $\mathcal{M}(X, Y)$. To construct a multiplication 2-cell for this monad, first notice that (traditional) matrix composition gives

$$\begin{pmatrix} 1_B + fg^*gf^* & fg^* \\ gf^* & 1_C \end{pmatrix}^2 = \begin{pmatrix} 1_B + 3\bullet fg^*gf^* + fg^*gf^*fg^*gf^* & 2\bullet fg^* + fg^*gf^*fg^* \\ 2\bullet gf^* + gf^*fg^*gf^* & 1_C + gf^*fg^* \end{pmatrix}$$

where

$$n\bullet h \overset{\text{def}}{=} \underbrace{h + h + \cdots + h}_{n \text{ terms}}$$

for $n \in \mathbb{N}$ and any 1-cell $h \in \mathcal{M}$. We provide a multiplication $\mu\colon m^2 \Rightarrow m$ "componentwise", by suplying a 2-cell between each pair of corresponding matrix entries. These can be constructed by combining various *fold* maps $\nabla\colon n \bullet h \Rightarrow h$ and the composite

$$\gamma = \; gf^*fg^* \xrightarrow[g\eta_f^{-1}g^*]{\simeq} gg^* \xrightarrow{\varepsilon_g} 1_C$$

(for which we need the assumption that f is fully faithful). For instance the first component is induced, via the universal property of $1_B + 3\bullet fg^*gf^* + fg^*gf^*fg^*gf^*$, by the 2-cells

$$\begin{gathered} 1_B \xrightarrow{c_0} 1_B + fg^*gf^* \\ 3 \bullet fg^*gf^* \xrightarrow{\nabla} fg^*gf^* \xrightarrow{c_1} 1_B + fg^*gf^* \\ fg^*gf^*fg^*gf^* \xrightarrow{fg^*\gamma gf^*} fg^*gf^* \xrightarrow{c_1} 1_B + fg^*gf^*, \end{gathered}$$

where c_0 and c_1 are the canonical coprojections associated with $1_B + fg^*gf^*$. The remaining components are defined as variations on this theme so we leave these details, along with the verification that (m, μ, ν) is a monad, up to the reader.

Consider the m-algebras into X. A 1-cell $a\colon B \oplus C \longrightarrow X$ is determined by a row vector (b, c) of 1-cells $b\colon B \longrightarrow X$ and $c\colon C \longrightarrow X$. Matrix multiplication gives

$$am = \begin{pmatrix} b & c \end{pmatrix} \begin{pmatrix} 1_B + fg^*gf^* & fg^* \\ gf^* & 1_C \end{pmatrix} = \begin{pmatrix} b + bfg^*gf^* + cgf^* & bfg^* + c \end{pmatrix}$$

so a 2-cell $\alpha\colon am \Rightarrow a$ is determined by giving 2-cells:

$$\begin{gathered} \alpha_0\colon b \Rightarrow b \quad \alpha_1\colon bfg^*gf^* \Rightarrow b \quad \alpha_2\colon cgf^* \Rightarrow b \\ \alpha_3\colon bfg^* \Rightarrow c \quad \alpha_4\colon c \Rightarrow c \end{gathered}$$

Unraveling the definition of η, we see that the unit condition on (a, α) reduces to the stipulation that $\alpha_0 = 1_b$ and $\alpha_4 = 1_c$. Compatibility with μ reduces to the commutativity of the following three diagrams:

(13) (i)
$$\begin{tikzcd} bfg^*gf^* \arrow[r, "\alpha_3 gf^*"] \arrow[dr, "\alpha_1"'] & cgf^* \arrow[d, "\alpha_2"] \\ & b \end{tikzcd}$$

(ii)
$$\begin{tikzcd} cgf^*fg^* \arrow[r, "\alpha_2 fg^*"] & bfg^* \arrow[d, "\alpha_3"] \\ cgg^* \arrow[u, "cg\eta_f g^*"] \arrow[r, "c\varepsilon_g"'] & c \end{tikzcd}$$

(iii)
$$\begin{tikzcd} bfg^*gf^*fg^*gf^* \arrow[rr, "\alpha_1 fg^*gf^*"] & & bf^*gf^* \arrow[d, "\alpha_1"] \\ bfg^*gg^*gf^* \arrow[u, "bfg^*g\eta_f g^*gf^*"] \arrow[r, "bfg^*\varepsilon_g gf^*"'] & bfg^*gf^* \arrow[r, "\alpha_1"'] & b \end{tikzcd}$$

The first of these eliminates α_1, so substitute for α_1 in (13)(iii) and simplify (by applying triangle identities and (13)(ii)). A little effort reveals that the commutativity of our pentagon, in the presence of the first two conditions, corresponds to the commutativity of:

$$
\begin{tikzcd}
bfg^*gf^* \arrow[r, "\alpha_3 gf^*"] & cgf^* \arrow[d, "\alpha_2"] \\
bff^* \arrow[u, "bf\eta_g f^*"] \arrow[r, "b\varepsilon_f"'] & b
\end{tikzcd}
\tag{14}
$$

Now α_2 and α_3 correspond, under the adjunctions $f \dashv f^*$ and $g \dashv g^*$, to 2-cells $\hat{\alpha}_2\colon cg \Rightarrow bf$ and $\hat{\alpha}_3\colon bf \Rightarrow cg$ respectively. Finally, by applying triangle identities, we may demonstrate that our remaining conditions (13)(ii) and (14) correspond to the identities $\hat{\alpha}_2\hat{\alpha}_3 = 1_{bf}$, $\hat{\alpha}_3\hat{\alpha}_2 = 1_{cg}$. Following the action of this construction on 2-cells, a process we leave up to the reader, it should be clear that we have constructed an equivalence:

$$\mathcal{M}(B\oplus C, X)^{\mathcal{M}(m,X)} \xrightarrow{\ \sim\ } \mathcal{M}(B,X) \underset{\mathcal{M}(A,X)}{\times} \mathcal{M}(C,X)$$

By assumption there is a Kleisli object D, presented by an m-algebra (r,ψ), and let $(h,k,\phi\colon hf \cong kg)$ be the corresponding cocone under (f,g), then we have a diagram

$$
\begin{tikzcd}
 & \mathcal{M}(D,X) \arrow[dl, "\simeq"'] \arrow[dr] \arrow[d, phantom, "\cong"] & \\
\mathcal{M}(B\oplus C, X)^{\mathcal{M}(m,X)} \arrow[rr, "\sim"] & & \mathcal{M}(B,X) \underset{\mathcal{M}(A,X)}{\times} \mathcal{M}(C,X)
\end{tikzcd}
$$

where the diagonals are induced by (r,ψ) and (h,k,ϕ) respectively. So the right hand diagonal is an equivalence as well, which means that (h,k,ϕ) is the pseudopushout of f along g in $\mathcal{M}$, and consequently in $\mathcal{M}^*$ (since factoring through the Kleisli object D respects maps), as required.

It is now easy to see that k is fully faithful, the monad m is isomorphic to that associated with $r \dashv r^*$, but $r^*r = \begin{pmatrix} h^* \\ k^* \end{pmatrix}\begin{pmatrix} h & k \end{pmatrix} = \begin{pmatrix} h^*h & h^*k \\ k^*h & k^*k \end{pmatrix}$. So the compatibility of the isomorphism $\begin{pmatrix} \theta_0 & \theta_1 \\ \theta_2 & \theta_3 \end{pmatrix}\colon \begin{pmatrix} 1_B+fg^*gf^* & fg^* \\ gf^* & 1_C \end{pmatrix} \cong \begin{pmatrix} h^*h & h^*k \\ k^*h & k^*k \end{pmatrix}$ with the units η and $\begin{pmatrix} \eta_h & 0 \\ 0 & \eta_k \end{pmatrix}$ implies that $\eta_k = \theta_3$, which is invertible. □

6.9. Theorem. *A map $e\colon A \longrightarrow B$ is Cauchy dense iff, for all fully faithful maps $j\colon X \longrightarrow Y$, all maps $u\colon A \longrightarrow X$, $v\colon B \longrightarrow Y$, and all invertible 2-cells $\tau\colon ju \cong ve$, there exists a map $w\colon B \longrightarrow X$ and an invertible 2-cell $\nu\colon jw \cong v$.*

Proof. "$\Rightarrow$" The map j is fully faithful, so we can form a composite isomorphism

$$\lambda = \quad u \xrightarrow[\cong]{\eta_j u} j^*ju \xrightarrow[\cong]{j^*\tau} j^*ve$$

which we use to make u into an m-algebra, where m is the monad associated with e, by supplementing it with a 2-cell:

$$\alpha = \quad ue^*e \xrightarrow[\cong]{\lambda e^* e} j^*vee^*e \xrightarrow{j^*v\varepsilon_e e} j^*ve \xrightarrow[\cong]{\lambda^{-1}} u$$

But e is Cauchy dense, which means that $(e, \varepsilon_e e)$ presents B as the Kleisli object of m, therefore there is an essentially unique 1-cell $w\colon B \longrightarrow X$ such that $w(e, \varepsilon_e e) \cong (u, \alpha)$. Furthermore Kleisli objects in $\mathcal{M}$ respect maps so w is a map since u is.

Notice that α is defined to ensure that the 2-cell $\tau\colon ju \cong ve$ gives an isomorphism of m-algebras (into Y) between $j(u, \alpha) = (ju, j\alpha)$ and $v(e, \varepsilon_e e) = (ve, v\varepsilon_e e)$. It follows that $jw(e, \varepsilon_e e) \cong j(u, \alpha) \cong v(e, \varepsilon_e e)$, or in other words the maps jw and v are obtained by factoring the same m-algebra through the limiting algebra $(e, \varepsilon_e e)$, so the essential uniqueness of such factorings implies that $jw \cong v$ as required.

"$\Leftarrow$" Consider the condition on e in the statement of the theorem. From the fact that j is fully faithful we can infer more about the induced map $w\colon B \longrightarrow Y$. For instance, if $w'\colon B \longrightarrow Y$ is another map along with an isomorphism $\nu'\colon jw' \cong v$, then the composite

$$w \xrightarrow[\cong]{\eta_j w} j^* jw \xrightarrow[\cong]{j^*\nu} j^* v \xrightarrow[\cong]{j^*(\nu')^{-1}} j^* jw' \xrightarrow[\cong]{\eta_j^{-1} w'} w'$$

demonstrates that $w \cong w'$. Reasoning similarly, the composite

$$we \xrightarrow[\cong]{\eta_j we} j^* jwe \xrightarrow[\cong]{j^*\nu e} j^* ve \xrightarrow[\cong]{j^*\tau} j^* ju \xrightarrow[\cong]{\eta_j^{-1} u} u$$

provides an isomorphism $\xi\colon we \cong u$, which satisfies the pasting identity:

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{\tau}{\cong}"] & B \arrow[d, "v"] \\ X \arrow[r, "j"'] & Y \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] \arrow[r, phantom, shift right=4, "\overset{\xi}{\cong}"] & B \arrow[d, "v"] \arrow[dl, "w"'] \\ X \arrow[r, "j"'] \arrow[r, phantom, shift left=4, "\overset{\nu}{\cong}"] & Y \end{tikzcd}$$

So suppose that e is a map satisfying the condition in the statement of this theorem. We know that there is a factorisation $e \cong ju$, with $j\colon C \longrightarrow B$ fully faithful and $u\colon A \longrightarrow C$ Cauchy dense, so setting $v = 1_B$ we may "fill the diagonal" to get a map $w\colon B \longrightarrow C$ and isomorphisms $\nu\colon jw \cong 1_B$, $\xi\colon we \cong u$. But the composite

$$wj \xrightarrow[\cong]{\eta_j kj} j^* jwj \xrightarrow[\cong]{j^*\nu j} j^* j \xrightarrow[\cong]{\eta_j^{-1}} 1_C$$

demonstrates that $wj \cong 1_C$, so j is an equivalence with inverse w. Therefore e, as a composite of this equivalence and the Cauchy dense map u, is itself Cauchy dense. □

6.10. Proposition 6.8 gives examples of pseudopushouts of maps which exist in any finitary cosmos, but in order to get all such pseudopushouts we need to adopt extra assumptions. An example of a finitary cosmos without pseudopushouts of maps is the bicategory of categories and profunctors internal to any elementary topos which lacks a natural numbers object.

6.11. If $r\colon A \longrightarrow A$ is an endo-1-cell on A then an *r-algebra into X*, (a, θ), consists of a 1-cell $a\colon A \longrightarrow X$ and a 2-cell $\theta\colon ar \Rightarrow a$. A morphism $\xi\colon (a,\theta) \Rightarrow (a',\theta')$ of r-algebras into X consists of a 2-cell $\xi\colon a \Rightarrow a'$ such that $\xi \cdot \theta = \theta' \cdot \xi r$. For each X we get a category $\mathrm{Alg}(\mathcal{M})(r, X)$ of r-algebras into X, and their morphisms.

An *algebraically free monad* on an endo-arrow $r\colon A \longrightarrow A$ is a monad m on A and a 2-cell $\lambda\colon r \Rightarrow m$ such that, for each 0-cell X, the functor

$$\tag{15}\begin{array}{ccc} \mathcal{M}(A,X)^{\mathcal{M}(m,X)} & \xrightarrow{\quad\tilde{\lambda}\quad} & \mathrm{Alg}(\mathcal{M})(r,X) \\ (a,\alpha) & \longmapsto & (a, \alpha \cdot a\lambda) \\ \xi\big\downarrow & \longmapsto & \big\downarrow\xi \\ (a',\alpha') & \longmapsto & (a', \alpha' \cdot a'\lambda) \end{array}$$

is an isomorphism. This condition is not really as strong as it might seem, if $\tilde{\lambda}$ is an equivalence it is easily demonstrated that it must necessarily be an isomorphism.

We define an *iterative cosmos* to be a finitary cosmos in which each endo-1-cells has an algebraically free monad upon it.

6.12. In a finitary cosmos, the algebraically free monad m upon an endo-1-cell $r\colon A \longrightarrow A$ is also free, in the sense that if n is another monad on A, and $\xi\colon r \Rightarrow n$ is any 2-cell, then there is a unique monad morphism $\phi\colon m \Rightarrow n$ with $\phi \cdot \lambda = \xi$.

To prove this, first fix algebras (a, α), presenting K as a Kleisli object for m, and (b, β), presenting L as a Kleisli object for n. Then $(b, \beta \cdot b\xi)$ is an r-algebra, so there is an m-algebra (b, γ) such that $\gamma \cdot b\lambda = \beta \cdot b\xi$ and, since (a, α) is Kleisli, there is an (essentially unique) map $c\colon K \longrightarrow L$ such that $c(a, \alpha) \cong (b, \gamma)$. But the adjoint transposes of $\alpha\colon am \Rightarrow a$, $\beta\colon bn \Rightarrow b$ provide us with isomorphisms $m \cong a^*a$, $n \cong b^*b$, so define ϕ to be the composite:

$$m \xrightarrow{\ \simeq\ } a^*a \xrightarrow{\ a^*\eta_c a\ } a^*c^*ca \xrightarrow{\ \simeq\ } b^*b \xrightarrow{\ \simeq\ } n$$

We leave it up to the reader to check that this is indeed a monad morphism, and the unique such with $\xi = \phi \cdot \lambda$.

6.13. An endo-1-cell $r\colon A \longrightarrow A$ together with a 2-cell $\rho\colon 1_A \Rightarrow r$ is called a *pointed endo-1-cell on A*. An (r, ρ)-algebra into X is an r-algebra (a, θ) into X satisfying $\theta \cdot a\rho = 1_a$, and there is a category $\mathrm{Alg}(\mathcal{M})((r,\rho), X)$ of (r, ρ)-algebras into X. We say that a monad m on A is the algebraically free monad on (r, ρ) if there is a 2-cell $\lambda\colon r \Rightarrow m$, with $\lambda \cdot \rho = \eta$, such that the obvious functor

$$\mathcal{M}(A,X)^{\mathcal{M}(m,X)} \xrightarrow{\quad\tilde{\lambda}\quad} \mathrm{Alg}(\mathcal{M})((r,\rho),X)$$

is an isomorphism for each 0-cell X. In a finitary cosmos $\mathcal{M}$ it is again true that an algebraically free monad on a pointed endo-1-cell is free in the usual sense.

If a pointed endo-1-cell (r, ρ) satisfies the extra condition $r\rho = \rho r$ then it is called *well pointed* (cf. [10]).

6.14. Remark. Suppose that (m, μ, η) and (n, κ, τ) are monads on $A \in \mathcal{M}$, and the 2-cells $\phi_0, \phi_1 : n \Rightarrow m$ are monad morphisms (that is, they are compatible with the unit and multiplication of n and m). Form their coequaliser $\beta: m \Rightarrow t$ in $\mathcal{M}(A, A)$, then the compatibility of ϕ_0, ϕ_1 with the multiplications of n and m ensures that we get a serially commutative diagram:

$$\begin{tikzcd}
nn \arrow[r, shift left, "\phi_0\phi_0"] \arrow[r, shift right, "\phi_1\phi_1"'] \arrow[d, "\kappa"'] & mm \arrow[r, "\beta\beta"] \arrow[d, "\mu"] & tt \arrow[d, dashed, "\sigma"] \\
n \arrow[r, shift left, "\phi_0"] \arrow[r, shift right, "\phi_1"'] & m \arrow[r, "\beta"'] & t
\end{tikzcd}$$

The preservation of such local coequalisers by composition in $\mathcal{M}$ ensures that the top line of this diagram is a coequaliser, thus the existence of the induced (dashed) 2-cell σ. This, along with a unit $\beta \cdot \eta: 1_A \Rightarrow t$, makes t into a monad, and β becomes a morphism of monads $(m, \mu, \eta) \Rightarrow (t, \sigma, \beta \cdot \eta)$. In fact t is the coequaliser of ϕ_0, ϕ_1 in the category of monads on A.

Given any morphism of monads $\phi: n \Rightarrow m$, we get a functor

$$\begin{array}{ccc}
\mathcal{M}(A, X)^{\mathcal{M}(m,X)} & \xrightarrow{\quad\tilde{\phi}\quad} & \mathcal{M}(A, X)^{\mathcal{M}(n,X)} \\
(a, \alpha) & \longmapsto & (a, \alpha \cdot a\phi)
\end{array}$$

for each 0-cell X. An important property of the coequaliser constructed in the last paragraph is that the diagram

$$\text{(16)}\quad \mathcal{M}(A, X)^{\mathcal{M}(t,X)} \xrightarrow{\ \tilde{\beta}\ } \mathcal{M}(A, X)^{\mathcal{M}(m,X)} \underset{\widetilde{\phi_1}}{\overset{\widetilde{\phi_0}}{\rightrightarrows}} \mathcal{M}(A, X)^{\mathcal{M}(n,X)}$$

is a strict (2-categorical) equaliser of categories, for each X. This follows directly from the assumption that local coequalisers are preserved by composition in $\mathcal{M}$

6.15. Proposition. *For a finitary cosmos $\mathcal{M}$ the following are equivalent:*

(a) each endo-1-cell r admits a algebraically free monad thereon;
(b) each pointed endo-1-cell (r, ρ) admits an algebraically free monad thereon;
(c) each well-pointed endo-1-cell (r, ρ) admits an algebraically free monad thereon.

Proof. **(a)⇒(c)** Suppose that (r, ρ) is a well-pointed endo-1-cell on A (as in (c)), then (a) ensures that we may form an algebraically free monad m upon the endo-1-cell r, presented by a 2-cell $\lambda: r \Rightarrow m$. Also let $\bar{1}_A$ denote the algebraically free monad on the identity 1_A, as presented by $\nu: 1_A \Rightarrow \bar{1}_A$. Now m has two points, its unit $\eta: 1_A \Rightarrow m$ and the composite $\lambda \cdot \rho: 1_A \Rightarrow m$, so the freeness property of $\bar{1}_A$, as given in 6.12, implies that we get unique monad morphisms $\phi_0, \phi_1 : \bar{1}_A \Rightarrow m$ with $\phi_0 \cdot \nu = \eta$, $\phi_1 \cdot \nu = \lambda \cdot \rho$. Taking the coequaliser of ϕ_0 and ϕ_1 we get a monad (t, σ, ζ) and a map $\beta \cdot \lambda: r \Rightarrow t$, cf. 6.14.

Now we form a serially commutative diagram

$$\begin{tikzcd}
\mathcal{M}(A,X)^{\mathcal{M}(t,X)} \arrow[r, "\tilde{\beta}"] & \mathcal{M}(A,X)^{\mathcal{M}(m,X)} \arrow[r, shift left, "\widetilde{\phi_0}"] \arrow[r, shift right, "\widetilde{\phi_1}"'] \arrow[d, "\tilde{\lambda}"', "\cong"] & \mathcal{M}(A,X)^{\mathcal{M}(\bar{1}_A,X)} \arrow[d, "\cong"', "\tilde{\nu}"] \\
& \mathrm{Alg}(\mathcal{M})((r,\rho),X) \arrow[r, shift left, "F_0"] \arrow[r, shift right, "F_1"'] & \mathrm{Alg}(\mathcal{M})(1_A,X)
\end{tikzcd}$$

in which the upper line is the strict equaliser from diagram (16). To calculate the actions of F_0, F_1 we use the defining equations $F_i\tilde{\lambda} = \tilde{\nu}\widetilde{\phi_i} = \widetilde{\phi_i\nu}$, along with the fact that $\tilde{\lambda}$ is an isomorphism. The equalities that we used to define ϕ_0 and ϕ_1 imply that $\widetilde{\phi_0\nu} = \tilde{\eta}$ and $\widetilde{\phi_1\nu} = \widetilde{\lambda\cdot\rho}$, but if (a,α) is an m-algebra then $\alpha\cdot a\eta = 1_a$ so $\tilde{\eta}(a,\alpha) = (a,1_a)$, also $\widetilde{\lambda\cdot\rho}(a,\alpha) = (a,(\alpha\cdot a\lambda)\cdot a\rho)$. Now if (a,θ) is an r-algebra, there is a unique m-algebra (a,α) with $\theta = \alpha\cdot a\lambda$, and we get:

$$\begin{aligned}
F_0(a,\theta) &= F_0\tilde{\lambda}(a,\alpha) = (a,1_a)\\
F_1(a,\theta) &= F_1\tilde{\lambda}(a,\alpha) = (a,(\alpha\cdot a\lambda)\cdot a\rho)\\
&= (a,\theta\cdot a\rho)
\end{aligned}$$

The functor $\widetilde{\beta\cdot\lambda}$ provides us with an isomorphism between $\mathcal{M}(A,X)^{\mathcal{M}(t,X)}$ and the strict equaliser of F_0 and F_1, but our descriptions of these functors reveal that this is the full subcategory of r-algebras (a,α) with $1_a = \alpha\cdot a\lambda$, or in other words the category of (r,ρ)-algebras. This is precisely what is needed to prove that $\beta\cdot\lambda$ presents t as the algebraically free monad on the (well-)pointed endo-1-cell (r,ρ), cf. 6.13.

(c)$\Rightarrow$(b) Let (r,ρ) be a pointed endo-1-cell on A. Form the cone on r

$$\begin{tikzcd}
A \arrow[rr, "r"] \arrow[dr, "v"'] & \arrow[d, phantom, "\overset{\omega}{\Leftarrow}"] & A \arrow[dl, "u"] \\
& \langle r,A\rangle &
\end{tikzcd}$$

as in 6.5, and let $\gamma\colon vr \Rightarrow w$ be the coequaliser of the two 2-cells $ur \Rightarrow vr$ given by:

$$\begin{tikzcd}
ur \arrow[r, shift left, "u\rho r"] \arrow[r, shift right, "ur\rho"'] & urr \arrow[r, "\omega r"] & vs
\end{tikzcd}$$

Now we define $t\colon \langle r,A\rangle \longrightarrow \langle r,A\rangle$, via the universal property of $\langle r,A\rangle$, to be the essentially unique 1-cell with isomorphisms $\theta\colon tv\cong w$, $\phi\colon tu\cong v$ such that $\gamma\cdot r\phi = \theta\cdot t\omega$. This has a point $\tau\colon 1_A\Rightarrow t$ defined, (again) via the universal property of $\langle r,A\rangle$, by $\theta\cdot v\tau = \gamma\cdot v\rho$ and $\phi\cdot u\tau = \omega\cdot u\rho$. One easily sees that $t\tau = \tau t$, and the category of (t,τ)-algebras into X is equivalent to that of (r,ρ)-algebras into X. So an algebraically free monad on (t,τ) gives one on (r,ρ). Compare with [10]; §17.1, p.50.

(b)⇒(a) If $r\colon A \longrightarrow A$ then $t = 1_A + r\colon A \longrightarrow A$ becomes pointed by the coprojection $\tau\colon 1_A \Rightarrow t$; moreover, the category of r-algebras into X is equivalent to that of (t,τ)-algebras into X. So an algebraically free monad on (t,τ) gives one on r. □

6.16. The axioms for an iterative cosmos are chosen to be finitary, but if we are willing to allow ourselves some countable local colimits we may construct algebraically free monads in a straightforward manner. For instance if $\mathcal{M}$ has countable local coproducts (which, as usual, are assumed to be preserved by composition) then the algebraically free monad on an endo-1-cell $r\colon A \longrightarrow A$ can be constructed as $\sum_{n=0}^{\infty} r^n$.

An application of the last proposition is to showing that weaker infinitary assumptions suffice to ensure that a finitary cosmos is iterative. Suppose that $\mathcal{M}$ has local colimits indexed by the ordered set of natural numbers, then the algebraically free monad on any well-pointed endo-1-cell (r,ρ) can be obtained by taking the colimit of the chain:

$$1_A \xrightarrow{\ \rho\ } r \xrightarrow{\ r\rho\ } r^2 \xrightarrow{\ r^2\rho\ } r^3 \xrightarrow{\ r^3\rho\ } r^4 \xrightarrow{\ r^4\rho\ } \cdots$$

But Proposition 6.15 ensures that, in establishing that $\mathcal{M}$ is iterative, we need only check that well-pointed endo-1-cells have free monads.

There are, of course, many finitary cosmoi which are iterative without satisfying any infinitary conditions. For instance, by arguing along the lines of [8]; §6.4, we see that the finitary cosmos Prof($\mathcal{E}$), of categories and profunctors internal to any elementary topos $\mathcal{E}$ with natural numbers object, is iterative.

6.17. Proposition. *A finitary cosmos $\mathcal{M}$ is iterative iff the associated bicategory of maps $\mathcal{M}^*$ has pseudopushouts, and they are preserved by the inclusion $\mathcal{M}^* \longrightarrow \mathcal{M}$.*

Proof. "⇐" Let r be an endo-1-cell on A. First form the cone on r

$$\begin{tikzcd} A \arrow[rr, "r"] \arrow[dr, "v"'] & \arrow[d, phantom, "\overset{\omega}{\Leftarrow}"] & A \arrow[dl, "u"] \\ & \langle r, A\rangle & \end{tikzcd}$$

then take the pseudopushout

$$\begin{tikzcd} A \oplus A \arrow[r, "\nabla"] \arrow[d, "{(u,v)}"'] \arrow[dr, phantom, "\cong"] & A \arrow[d, "k"] \\ \langle r, A\rangle \arrow[r, "h"'] & P \end{tikzcd}$$

in $\mathcal{M}^*$, which is also a pseudopushout in $\mathcal{M}$ (since $\mathcal{M}^* \longrightarrow \mathcal{M}$ preserves pseudopushouts). So the category $\mathcal{M}(P,X)$ is equivalent to a category whose objects are pairs (p,τ) where $p\colon \langle r, A\rangle \longrightarrow X$ is a 1-cell and $\tau\colon pu \cong pv$ an invertible 2-cell. In turn, by using the universal property of the cone $\langle r, A\rangle$, we see that this latter category is equivalent to the category of r-algebras into X.

In 6.5 it was noted that $(u,v)\colon A \oplus A \longrightarrow \langle r, A\rangle$ is Cauchy dense, that is to say liberal in $\mathcal{M}$; but any pseudopushout of a liberal is again liberal, so k is Cauchy dense as well. Now 6.3 shows that if m is the monad associated with $k \dashv k^*$, then $(k, \varepsilon_k k)$ presents P as the Kleisli object of m. So, composing the consequent equivalence $\mathcal{M}(A,X)^{\mathcal{M}(m,X)} \simeq \mathcal{M}(P,X)$ with $\mathcal{M}(P,X) \simeq \mathrm{Alg}(\mathcal{M})(r,X)$ from the last paragraph, we get an equivalence which is easily shown to be induced by the obvious 2-cell $\lambda\colon r \Rightarrow m$. This is precisely what is required to prove that m is the algebraically free monad on r.

"$\Rightarrow$" To construct the pseudopushout of the pair of maps $f\colon A \longrightarrow B$, $g\colon A \longrightarrow C$; we first form the coproduct $B \oplus C$, and consider the endo-1-cell r upon it, determined by the matrix

$$r = \begin{pmatrix} 1_B & fg^* \\ gf^* & 1_C \end{pmatrix}$$

which has an obvious point:

$$\rho = \begin{pmatrix} 1_{1_B} & !_{fg^*} \\ !_{gf^*} & 1_{1_C} \end{pmatrix} \colon \begin{pmatrix} 1_B & 0 \\ 0 & 1_C \end{pmatrix} \longrightarrow \begin{pmatrix} 1_B & fg^* \\ gf^* & 1_C \end{pmatrix}$$

By assumption, and Proposition 6.15, there is a monad (m, μ, η) and a 2-cell $\lambda\colon r \Rightarrow m$ (with $\lambda \cdot \rho = \eta$) displaying m as the algebraically free monad on (r, ρ). Using the universal property of $B \oplus C$, it is easily shown that the category of (r,ρ)-algebras into X is equivalent to a category with objects consisting of a pair of 1-cells $b\colon B \longrightarrow X$, $c\colon C \longrightarrow X$, and an action $\theta\colon (b\ c)r \Rightarrow (b\ c)$ compatible with the point ρ. But $(b\ c)r = (b + cgf^*\ \ bfg^* + c)$, so θ corresponds to a family of 2-cells

$$\theta_0\colon b \Rightarrow b \qquad \theta_1\colon c \Rightarrow c \qquad \theta_2\colon cgf^* \Rightarrow b \qquad \theta_3\colon bfg^* \Rightarrow c$$

and compatibility with the point ρ corresponds to saying that $\theta_0 = 1_b$ and $\theta_1 = 1_c$. So the category of m-algebras, which is isomorphic to the category of (r,ρ)-algebras, is equivalent to a category of pairs $(b\ c)$ equipped with 2-cells $\hat{\theta}_2\colon cg \Rightarrow bf$, $\hat{\theta}_3\colon bf \Rightarrow cg$ (these correspond to θ_2, θ_3 under the adjunctions $f \dashv f^*$, $g \dashv g^*$). Somehow we must take a quotient of m, thereby imposing conditions ensuring that $\hat{\theta}_2$ and $\hat{\theta}_3$ are mutual inverses.

To this end, consider the endo-1-cell $r^2 = \begin{pmatrix} 1_B + fg^*gf^* & 2\bullet fg^* \\ 2\bullet gf^* & 1_C + gf^*fg^* \end{pmatrix}$ and the new matrix:

$$s = \begin{pmatrix} ff^* & 0 \\ 0 & gg^* \end{pmatrix}$$

There are two canonical 2-cells $\psi_0, \psi_1\colon s \Rightarrow r^2$, given by matrices $\begin{pmatrix} \psi_{i0} & ! \\ ! & \psi_{i1} \end{pmatrix}$ $(i = 0, 1)$, the first with non-trivial components

$$\psi_{00} = \ ff^* \xrightarrow{\ \varepsilon_f\ } 1_B \xrightarrow{\ c_0\ } 1_B + fg^*gf^*$$
$$\psi_{01} = \ gg^* \xrightarrow{\ \varepsilon_g\ } 1_C \xrightarrow{\ c_0\ } 1_C + gf^*fg^*$$

and the second with:

$$\psi_{10} = \ ff^* \xrightarrow{\ f\eta_g f^*\ } fg^*gf^* \xrightarrow{\ c_1\ } 1_B + fg^*gf^*$$
$$\psi_{11} = \ gg^* \xrightarrow{\ g\eta_f g^*\ } gf^*fg^* \xrightarrow{\ c_1\ } 1_C + gf^*fg^*$$

Let (n, κ, τ) be the algebraically free monad on s, displayed by a 2-cell $\nu\colon s \Rightarrow n$. By composing ψ_0, ψ_1 with the 2-cell $\mu \cdot \lambda^2\colon r^2 \Rightarrow m$, then applying the freeness property of n described in 6.12, we get monad morphisms $\phi_0, \phi_1\colon n \Rightarrow m$ such that $\phi_0 \cdot \nu = \mu \cdot \lambda^2 \cdot \psi_0$ and $\phi_1 \cdot \nu = \mu \cdot \lambda^2 \cdot \psi_1$.

Now, following Remark 6.14, we take the coequaliser of ϕ_0, ϕ_0 thereby obtaining a new monad t and a serially commutative diagram

$$\begin{tikzcd}
\mathcal{M}(A,X)^{\mathcal{M}(t,X)} \arrow[r, "\tilde{\beta}"] & \mathcal{M}(A,X)^{\mathcal{M}(m,X)} \arrow[r, shift left, "\widetilde{\phi_0}"] \arrow[r, shift right, "\widetilde{\phi_1}"'] \arrow[d, "\tilde{\lambda}"', "\cong"] & \mathcal{M}(A,X)^{\mathcal{M}(n,X)} \arrow[d, "\tilde{\nu}", "\cong"'] \\
& \mathrm{Alg}(\mathcal{M})((r,\rho),X) \arrow[r, shift left, "F_0"] \arrow[r, shift right, "F_1"'] & \mathrm{Alg}(\mathcal{M})(s,X)
\end{tikzcd}$$

in which the upper line is the strict equaliser from diagram (16). To work out the action of F_0 (resp. F_1) on (r, ρ)-algebras, consider the action of $\tilde{\nu}\widetilde{\phi_i} = F_i\tilde{\lambda}$ $(i = 0, 1)$ on an m-algebra (a, α):

$$\begin{aligned}
\tilde{\nu}\widetilde{\phi_i}(a,\alpha) &= (a, \alpha \cdot a(\phi_i \cdot \nu)) & \\
&= (a, \alpha \cdot a(\mu \cdot \lambda^2 \cdot \psi_i)) & \text{definition of } \phi_i \\
&= (a, \alpha \cdot \alpha m \cdot a\lambda^2 \cdot a\psi_i) & (a,\alpha) \text{ is an } m\text{-algebra} \\
&= (a, \alpha \cdot (\alpha \cdot a\lambda)m \cdot ar\lambda \cdot a\psi_i) & \\
&= (a, (\alpha \cdot a\lambda) \cdot (\alpha \cdot a\lambda)r \cdot a\psi_i) & \text{middle four interchange.}
\end{aligned}$$

But $\tilde{\lambda}(a, \alpha) = (a, \alpha \cdot a\lambda)$, and $\tilde{\lambda}$ is an isomorphism, so for each (r, ρ)-algebra (a, θ) there exists an m-algebra (a, α) with $\theta = \alpha \cdot a\lambda$, and therefore $F_i(a, \theta) = \tilde{\nu}\widetilde{\phi_i}(a, \alpha) = (a, (\alpha \cdot a\lambda) \cdot (\alpha \cdot a\lambda)r \cdot a\psi_i) = (a, \theta \cdot \theta r \cdot a\psi_i)$. It follows that the functor $\tilde{\lambda}\tilde{\beta}$ identifies $\mathcal{M}(A,X)^{\mathcal{M}(t,X)}$ with the full subcategory of $\mathrm{Alg}(\mathcal{M})((r,\rho),X)$, on those (r, ρ)-algebras (a, θ) with $\theta \cdot \theta r \cdot a\psi_0 = \theta \cdot \theta r \cdot a\psi_1$.

Recall that we have already re-expressed the category $\mathrm{Alg}(\mathcal{M})((r,\rho),X)$ (up to equivalence) by using the universal property of $B \oplus C$. For a pair of 1-cells $b\colon B \longrightarrow X$, $c\colon C \longrightarrow X$ we have

$$\begin{aligned}
(b\ c)r^2 &\cong (b\ c)\begin{pmatrix} 1_B + fg^*gf^* & 2 \bullet fg^* \\ 2 \bullet gf^* & 1_c + gf^*fg^* \end{pmatrix} \\
&\cong (b + bfg^*gf^* + 2 \bullet cgf^* \quad 2 \bullet bfg^* + c + cgf^*fg^*)
\end{aligned}$$

and the 2-cell $\theta \cdot \theta r =: (b\ c)r^2 \Rightarrow (b\ c)$ may be given in terms of six components:

$$b \xrightarrow{\quad 1_b \quad} b \qquad bfg^*gf^* \xrightarrow{\theta_3 gf^*} cgf^* \xrightarrow{\theta_2} b$$

$$2 \bullet cgf^* \xrightarrow{\nabla} cgf^* \xrightarrow{\theta_2} b \qquad 2 \bullet bfg^* \xrightarrow{\nabla} bfg^* \xrightarrow{\theta_3} c$$

$$cgf^*fg^* \xrightarrow{\theta_2 fg^*} bfg^* \xrightarrow{\theta_3} c \qquad c \xrightarrow{\quad 1_c \quad} c$$

Similarly $(b\ c)s = (bff^*\ cgg^*)$, so an s-algebra structure on $(b\ c)$ consists of a pair of 2-cells $bff^* \Rightarrow b$, $cgg^* \Rightarrow c$. Finally, putting all this together with the definitions of ψ_0 and ψ_1, we see that the equation $\theta \cdot \theta r \cdot a\psi_0 = \theta \cdot \theta r \cdot a\psi_1$ reduces to the commutativity of diagrams:

$$\begin{tikzcd} bfg^*gf^* \arrow[r, "\theta_3 gf^*"] & cgf^* \arrow[d, "\theta_2"] \\ bff^* \arrow[u, "bf\eta_g f^*"] \arrow[r, "b\varepsilon_f"'] & b \end{tikzcd} \qquad \begin{tikzcd} cgf^*fg^* \arrow[r, "\theta_2 fg^*"] & bfg^* \arrow[d, "\theta_3"] \\ cgg^* \arrow[u, "cg\eta_f g^*"] \arrow[r, "c\varepsilon_g"'] & c \end{tikzcd}$$

Compare these with conditions (13)(ii) and (14) in (the proof of) Proposition 6.8. The rest is identical to that proof, our conditions correspond to the identities $\hat{\theta}_2\hat{\theta}_3 = 1$ and $\hat{\theta}_3\hat{\theta}_2 = 1$, and we get an equivalence

$$\mathcal{M}(B \oplus C, X)^{\mathcal{M}(t,X)} \xrightarrow{\sim} \mathcal{M}(B, X) \underset{\mathcal{M}(A,X)}{\times} \mathcal{M}(C, X)$$

from which we infer that the Kleisli object of the monad t provides us with a pseudopushout of f along g. □

Finally we come to the principal result of this section:

6.18. Theorem. *If $\mathcal{M}$ is an iterative cosmos then $\mathcal{M}^*$ is a modulated bicategory.*

Proof. In showing that $\mathcal{M}^*$ is modulated we must first show that $(\mathcal{M}^*)^{\mathrm{op}}$ is faithfully conservational (cf. definitions 2.18, 2.25). Recall the comments in 6.5; tensors with $\mathbf{2}$ and terminal objects are both examples of collages, so $\mathcal{M}$ possesses these. But collages respect maps so these are also the corresponding colimits in $\mathcal{M}^*$. Proposition 6.17 provides $\mathcal{M}^*$ with pseudopushouts, which are also preserved by $\mathcal{M}^* \longrightarrow \mathcal{M}$, so Theorem 1.14 implies that $\mathcal{M}^*$ has all finite colimits and that they are preserved by the inclusion $\mathcal{M}^* \longrightarrow \mathcal{M}$. This serves to establish 2.18(i).

The truth of the remaining conditions hinge on identifying the classes of liberals and strong conservatives in $\mathcal{M}^*$. Following 2.3 we know that a map $e\colon A \longrightarrow B$ is liberal iff the following square is a pseudopushout in $\mathcal{M}^*$.

$$\begin{tikzcd} \mathbf{2} * A \arrow[r, "\nabla"] \arrow[d, "\mathbf{2} * e"'] \arrow[dr, phantom, "\cong"] & A \arrow[d, "e"] \\ \mathbf{2} * B \arrow[r, "\nabla"'] & B \end{tikzcd}$$

But all finite colimits are preserved by $\mathcal{M}^* \longrightarrow \mathcal{M}$, so this diagram is mapped to the corresponding one in $\mathcal{M}$, and this is a pseudpushout in $\mathcal{M}$ iff it is so in $\mathcal{M}^*$. In other words the liberal 1-cells in $\mathcal{M}^*$ are precisely those maps which are liberal as 1-cells in $\mathcal{M}$; these are in turn exactly the Cauchy dense maps (by 6.3).

We know that $\mathcal{M}^*$ has tensors with $\mathbf{2}$, so for a map to be strong conservative it need only satisfy (the appropriate dual of) of the "fill in" property of 2.13, with respect to the Cauchy dense maps. Theorem 6.9, and comments contained in its proof, demonstrate that fully faithful maps are certainly strong conservatives, but suppose conversely that $f\colon A \longrightarrow B$ is a strong conservative map. We know that it factors as $\tau\colon f \cong je$ with $e\colon A \longrightarrow B$ Cauchy dense and $j\colon B \longrightarrow C$ fully faithful, so the diagonal "fill in" property, for f relative to e, gives w, ξ and ν in:

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\overset{\tau}{\cong}"] & B \arrow[d, "j"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "e"] \arrow[d, "1_A"'] & B \arrow[d, "j"] \arrow[dl, "w"'] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

with $\overset{\xi}{\cong}$ in the upper triangle and $\overset{\nu}{\cong}$ in the lower triangle.

But j is fully faithful, so we have the composite isomorphism

$$ew \xrightarrow[\cong]{\eta_j ew} j^*jew \xrightarrow[\cong]{j^*\tau^{-1}w} j^*fw \xrightarrow[\cong]{j^*\nu} j^*j \xrightarrow[\cong]{\eta_j^{-1}} 1_B$$

which, along with ξ, establishes that w is an equivalence inverse to e. Since f is a composite of a fully faithful map and an equivalence, it is itself fully faithful. Our factorisation of maps, into a fully faithful following a Cauchy dense, now becomes that satisfying axiom 2.18(iv).

Proposition 6.8, which constructed pseudopushouts of fully faithful maps, shows that axiom 2.18(ii) holds; meanwhile 6.7 demonstrates that if j is fully faithful so is $\mathbf{2} * j$, as required by 2.18(iii). Cauchy dense maps are clearly faithful 1-cells in $(\mathcal{M}^*)^{\mathrm{op}}$, which completes the verification that this bicategory is faithfully conservational.

All that remains for us is to check the modulation axioms of 5.2. The first is easy, since the cocomma object of a pair of maps $f\colon A \longrightarrow B$ and $g\colon A \longrightarrow C$ may be constructed as the collage of the 1-cell $gf^*\colon B \longrightarrow C$, and we observed in 6.5 that the coprojections into such a collage are fully faithful.

Finally we verify 5.2(vi), let $a = (a_1\ a_2\ a_3)\colon A_1 \oplus A_2 \oplus A_3 \longrightarrow X$ be Cauchy dense and $j_1(e_{12}\ e_{13}) \cong (a_2\ a_3)$, $j_2(e_{21}\ e_{23}) \cong (a_1\ a_3)$ and $j_3(e_{31}\ e_{32}) \cong (a_1\ a_2)$ be the required factorisations into fully faithful maps following Cauchy dense ones. Suppose now that $f\colon X \longrightarrow K$ is a map such that fj_m is fully faithful for $m = 1, 2, 3$. Since a is Cauchy dense, the map f is determined by the a^*a-algebra structure on $fa \cong (fa_1\ fa_2\ fa_3)\colon A_1 \oplus A_2 \oplus A_3 \longrightarrow K$ which has adjoint transpose (under $fa \dashv a^*f^*$) given by the 3×3 matrix:

$$\left(a_k^*\eta_f a_l\right)_{k,l}\colon \left(a_k^* a_l\right)_{k,l} \longrightarrow \left(a_k^* f^* f a_l\right)_{k,l}$$

For each pair $1 \le k, l \le 3$ pick $1 \le m \le 3$ with $m \ne k, l$; then $a_l \cong j_m e_{ml}$ and $a_k^* \cong e_{mk}^* j_m^*$ so it follows that $a_k^*\eta_f a_l$ is invertible iff $e_{mk}^* j_m^* \eta_f j_m e_{ml}$ is. Of course both j_m and fj_m (which has unit $j_m^*\eta_f j_m \cdot \eta_{j_m}$) are fully faithful, by assumption, so $j_m^*\eta_f j_m$ is an isomorphism. In summary each component of our matrix is invertible, implying that the 2-cell it represents is also invertible, and therefore we can apply 6.6 to infer that f itself is fully faithful. □

6.19. Of particular interest is the special case wherein we restrict our bicategories to be *locally ordered*, meaning that each homcategory is a partially ordered set (which we consider to be a category in the usual way). A bicategory $\mathcal{H}$ is called *idyllic* when it is modulated, locally ordered and each liberal arrow is pseudoepic.

Of course we should provide examples of idyllic bicategories; to fill in the details of the following construction we refer the reader to [6]. Let $\mathcal{E}$ be a regular category, there is a bicategory $\mathrm{Idl}(\mathcal{E})$ whose objects are the (partially) ordered objects in $\mathcal{E}$, whose arrows are ideals, and whose local ordering is by "inclusion" of ideals. The composite of two ideals is that customary for relations. We also have a locally ordered bicategory $\mathrm{Ord}(\mathcal{E})$ with the same objects, but with arrows which are order preserving functions, which are themselves ordered "pointwise". Finally, there is a canonical homomorphism $(\ )_*\colon \mathrm{Ord}(\mathcal{E})\longrightarrow\mathrm{Idl}(\mathcal{E})$, which acts as the identity on objects; the ideal f_* has a right adjoint f^* for each 1-cell $f \in \mathrm{Ord}(\mathcal{E})$.

If $\mathcal{E}$ is an elementary topos with natural numbers object then it is easily demonstrated that $\mathrm{Idl}(\mathcal{E})$ is an iterative cosmos. The finite colimits of $\mathcal{E}$ provide local finite colimits and global finite bicategorical coproducts. A monad R on the ordered object $(A, \leq)$ is precisely a transitive relation on A containing $\leq$. So (A, R) is a 0-cell of $\mathrm{Idl}(\mathcal{E})$ and the identity on A is an order preserving function $(A, \leq)\longrightarrow(A, R)$; the associated adjunction of ideals presents (A, R) as the Kleisli object of I. Finally the algebraically free monad upon an endo-ideal I on $(A, \leq)$ is simply the smallest transitive relation on A containing both $\leq$ and I, which may be constructed by recursion as in [8] Section 6.3.

We now apply Theorem 6.18 and thereby infer that the category of maps $\mathrm{Idl}(\mathcal{E})^*$ is a modulated bicategory. Since $\mathrm{Idl}(\mathcal{E})$ is locally ordered the Cauchy dense maps in $\mathrm{Idl}(\mathcal{E})$, which are the liberal arrows of $\mathrm{Idl}(\mathcal{E})^*$, are precisely those with counit an equality. So these are split epics in $\mathrm{Idl}(\mathcal{E})$, which implies that they are pseudoepic, both in there and in $\mathrm{Idl}(\mathcal{E})^*$. This completes the proof that $\mathrm{Idl}(\mathcal{E})^*$ is idyllic.

Infact, $\mathrm{Ord}(\mathcal{E})$ is also idyllic with $\mathrm{DFib}(\mathrm{Ord}(\mathcal{E})^{\mathrm{op}})^{\mathrm{coop}} = \mathrm{Idl}(\mathcal{E})$. However (see [6] Corollary 4), we have $\mathrm{Ord}(\mathcal{E}) = \mathrm{Idl}(\mathcal{E})^*$ iff $\mathcal{E}$ satisfies the axiom of choice.

6.20. Theorem 6.18 allows us to apply the constructions of previous sections to $\mathcal{M}^*$ for any iterative cosmos $\mathcal{M}$. In fact [22], which deals specifically with enriched categories, and [19], which concerns a generalisation to a cosmos-like setting, show that the bicategory $\mathrm{DFib}(\mathcal{M}^*)$ is canonically equivalent to the dual $\mathcal{M}^{\mathrm{coop}}$. The novelty in our approach is the introduction of a factorisation system, with which to make sense of the composition of discrete fibrations. Earlier work used equinverters for this task, not altogether successfully.

Following the work of the last section we might like to prove a converse to the last theorem. In other words, starting with a comodulated bicategory $\mathcal{K}$ we ask ourselves the question "is $\mathrm{DFib}(\mathcal{K})^{\mathrm{coop}}$ an iterative cosmos?" Our work has already made a step in this direction, by establishing that $\mathrm{DFib}(\mathcal{K})^{\mathrm{coop}}$ has local finite colimits and collages of 1-cells. It seems unlikely however that the answer to our question will be in the affirmative, with the principal task becoming that of adding axioms on $\mathcal{K}$ to force the existence of free monads and more general collages in $\mathrm{DFib}(\mathcal{K})^{\mathrm{coop}}$.

A related question, which we have not discussed here, is that of determining the bicategory of maps in $\mathrm{DFib}(\mathcal{K})$. Ideally we would like this to be $\mathcal{K}$ itself, which it certainly contains, but again we may need further axioms to force equality.

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